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On Stress Analysis of a Crack-Layer

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LIST OF SYMBOLS

ρ - damage density

ρ_o - critical level of damage density

V_A - crack-layer active zone

V_I - crack-layer inert zone

J, L, M - the crack-layer driving forces

ℓ_i - the half length of i-th crack

i, j, k, n, m, s - summation indexes, n, k - parameters of elliptic integrals

K_I^{eff} - mode I effective stress intensity factor

K_I^o - mode I stress intensity factor of the main crack

K_{II}^{eff} - mode II effective stress intensity factor

$\underset{\sim}{n}$ - unit normal vector

$\underset{\sim}{X}$ - position vector

\- stress tensor

N - number of microcracks in the array

$\underset{\sim}{\phi}$ - angle distribution tensor in the asymtotic stress field of the main crack

$\underset{\sim}{r}$ - position vector in the crack tip coordinate system

θ - polar angle in the crack tip coordinate system

b - double layer potential density

$\underset{\sim}{\Phi}$ - second Green's tensor for infinite elastic plane

$\underset{\sim}{\xi}$ - position vector

ν - Poisson's ratio

E - modulus of elasticity

$\underline{E}$ - unit tensor

$\underset{\sim}{R}$ - vector connecting two points in an elastic plane

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$\underset{\sim}{T}_x$ - stress operator

C - the distance between the centers of the macro and micro-cracks

δ - the distance between the crack tips

δ_1, δ_2 - the horizontal and vertical distances between the crack tips, respectively

$e(\xi)$ - elliptic crack opening displacement

α,β,γ - tensor indexes, constants

q - the function which determines the effective stress intensity factor

ℓ', C', δ' - nondimensionalized distances

f_1, f_2, f - some functions

h - the vertical distance between two symmetrically located microcracks

F - the influence function

$\{b\}: b_i$ - the coefficients of a crack opening displacement polynomial

$\{\underset{\sim}{\sigma}\}: \sigma', \sigma'', \sigma^{(K)}$ - consequent derivatives of the stress tensor in the direction of the microcrack

q_i - the functions determing the corrections to the effective stress intensity factor due to i-th order polynomial approximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{X})$

I_i - the integrals

$F(V)$, (N,K), $I(n,K)$ - the complete elliptic integrals of the first and third kind and the integral of similar structure, respectively.

$\{B\}$ - transformation matrix, $\{B\} = B_{mn}(X_o^S, X_o^K)$ - transformation operator

S : $S'(X)$, $S''(X)$, $S^{(K)}(X)$ - consequent derivatives of the crack tip singularity function

P - the degree of a stress field approximating polynomial

C_n^m - bynomial coefficients

$\sigma_0(x)$ - asymtotic stress field of a macrocrack

$\{t\}$: t_1, t_2, ..., t_m - consequent derivatives of a traction vector in the direction of the microcrack

$\underset{\sim}{A}(\underset{\sim}{x})$ - matrix of the microcrack array

$\{A^0\}$ - linear operator

$\Gamma, \Gamma_0, \Gamma_1, F_2$ - contours of integration

ΔJ - increment of J-integral increase

$\{\underset{\sim}{H}\}$, $\{\underset{\sim}{I}\}$, $\{\underset{\sim}{D}\}$, $\{\underset{\sim}{1}\}$ - linear operators

$\underset{\sim}{0}$ - unit force vector

$\underset{\sim}{U}$ - Kelvin-Somigliana second rank tensor

CHAPTER I

Micromechanics of a Crack-Layer (CL)

1. Introduction

In recent years a significant amount of experimental data has been accumulated on a process zone surrounding the tip of a propagating crack [1,2,3,4,5,6]. Process zone is usually defined as the area of severely damaged material adjacent to the crack tip. It has been shown in works [7,8,9,10] that mophology of damage zone varies from one material to the other. In these works ceramics, rocks, polymers, and metals were investigated. It was found that damage constituting the process zone can reveal itself as microcracking [3, 11, 12,6] in all of the above materials; martensitic transformation in ceramics, metals, and polymers [13,14,15,7], slip lines (i.e., shear bands) in metals and polymers [16,17], crazing in polymers, etc.

Despite the difference in morphology of process zones in various materials, there are similar features in all of them. For example, similar global geometry and similar kinetics of development have been observed [18]. Theoretical models have been proposed for the description of kinetics of a process zone [9,13,16,18,19]. It should be noted that traditional fracture mechanics can be considered as one of them. Both linear and nonlinear fracture mechanics take into consideration the crack tip plastic zone and use the well developed techniques of plasticity theory for estimates of its size, shape, etc. Damage distribution, however, can be quite different from the

FATIGUE CRACK AND SURROUNDING DAMAGE IN POLYPROPYLENE.

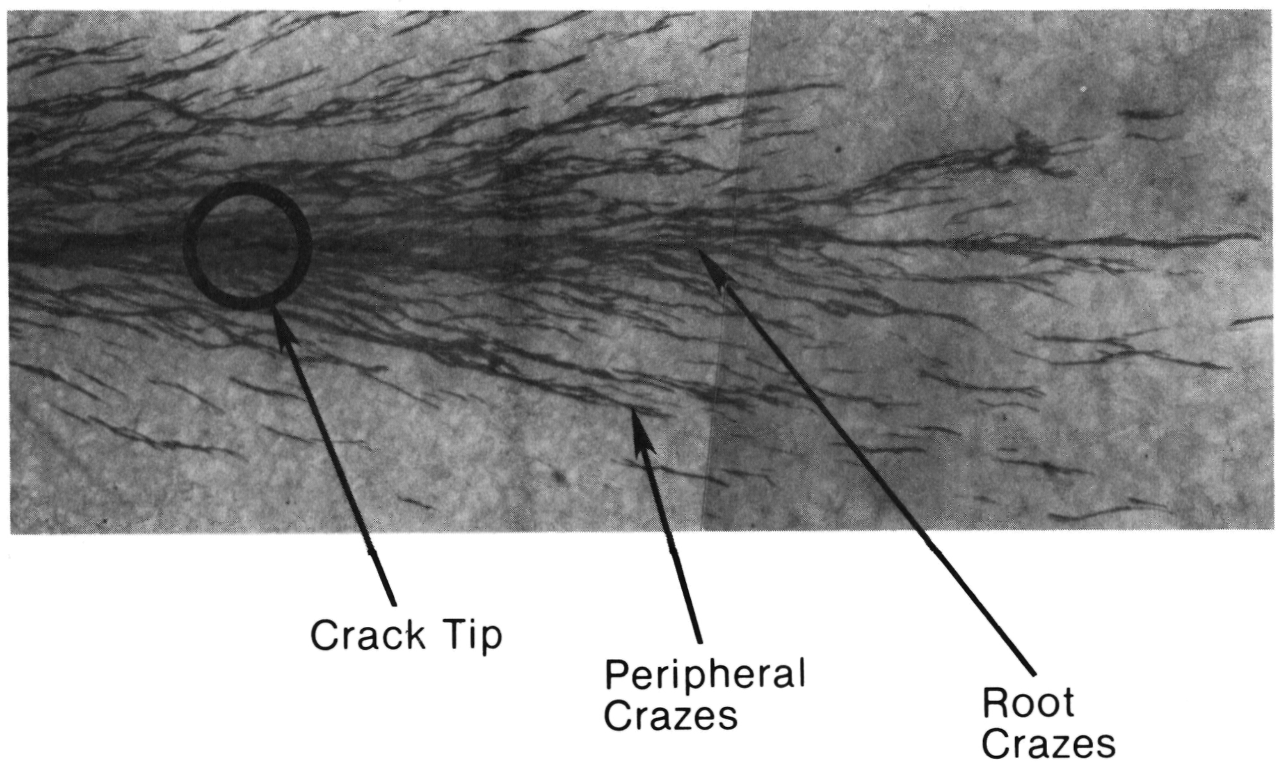

FIGURE 1

FATIGUE CRACK AND SURROUNDING DAMAGE IN RIGID PVC.

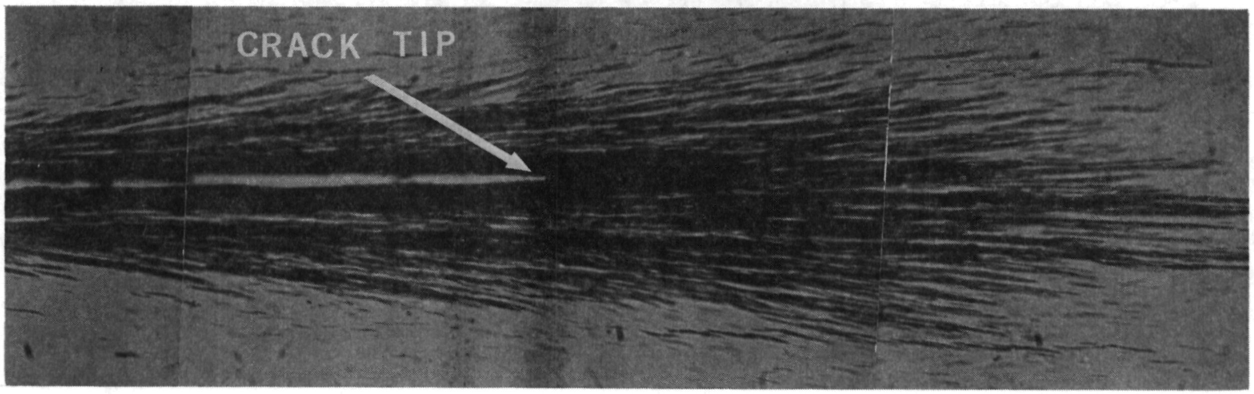

FIGURE 2

FATIGUE CRACK AND SURROUNDING DAMAGE IN STAINLESS STEEL.

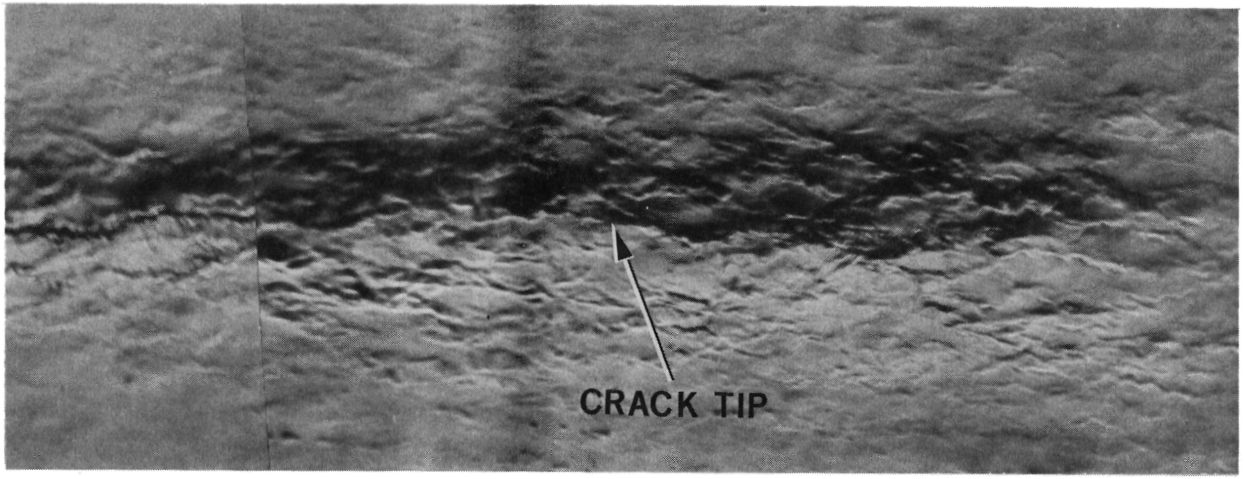

FIGURE 3

one described by models of plastic behavior. The results of work [20] clearly demonstrate that some damage patterns do not yield any model of plasticity. The theoretical model of a crack layer (CL), as opposed to the crack-cut of traditional fracture mechanics, was proposed by A. Chudnovsky in 1976 [18]. In the CL theory a crack with the surrounding damage is considered as a single macroscopic entity. The process of crack propagation is described as a nucleation, development, and subsequent coalescence of microdefects in a crack tip zone. Recently the theory was examined in works [20-22] with the goal to observe damage distribution patterns in a crack tip zone and to investigate the kinetics of damage development. This has been done for various materials and different loading histories. Figures [1,2,3] taken from [20,21] represent microphotographs which illustrate the discussion above. These experimental results lead to the schematic representation of a CL on Figure 4.

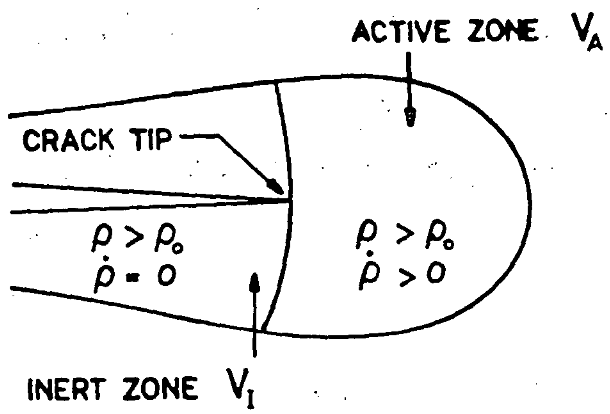

Figure 4

The CL is represented as a crack-cut surrounded by damage of density ρ. Damage density ρ exceeds the level ρ_0 (i.e., $\rho > \rho_0$) in both macrocrack tip active and inert zones; damage density rate ρ, however, is positive in active zone V_A, but vanishes in the inert zone V_I.

Instead of a detailed description of damage density ρ and its evolution, the CL theory operates with integral characteristics of the damage zone. These integral characteristics are: the length, the width and the shape of the active zone. The main crack length and the curvature of the crack trajectory also enter as geometric characteristics of a CL (we talk of crack trajectory rather than crack surface when considering the crack within the framework of a plane problem). On the basis of general principles of irreversible thermodynamics, the authors of works [18,23,24] have introduced CL driving forces which are reciprocal to geometric parameters of a CL. The detailed description of a CL geometry and kinetics can be found in [25]. This work introduces four CL driving forces which appear to be reciprocals to 1. CL length, 2. the curvature of CL trajectory, 3. active zone area, and 4. active zone shape (active zone length to width ratio). The first three CL driving forces are represented as linear combinations of the well known path-independent integrals J, L and M of fracture mechanics and, the integral N, [25] which is not path-independent in general.

The integrals J, L, and M appeared in the formulation of conservation laws of elastostatics, (i.e. $J = 0$, $L = 0$, and $M = 0$ on any closed contour without singularities inside of it, [26]). These conservation laws have been derived by means of Noether's theorem from the principle of minimum of strain energy. The conservation laws $J = 0$, $L = 0$, and $M = 0$ result from invariance of strain energy functional with respect to the group of displacements, rotations, and infinitesimal isotropic expansions, respectively. Naturally, the first one holds for homogeneous material only, the second holds only for isotropic and the third one for linear medium. The conservation law involving J-integral appeared for the first time in work [27] of 1951 by Eshelby. In terms of the J-integral Eshelby expressed the force acting on a singularity in an elastic body. Later, the J-integral was rederived by Sanders [28] in 1960, Cherepanov in 1967 [29], and Rice in 1968 [30] in connection with a problem of energy release rate in a quasistatic crack propagation process. The tangential to the crack component of vector J was found to be precisely the energy release rate. After the discovery of the other two conservation laws, L and M were interpreted as energy release rates with respect to cavity rotation and cavity expansion, respectively [31]. In works [23,24,25], by A. Chudnovsky et al, the path-independent integrals appeared as parts of thermodynamic crack driving forces.

For evaluation of J, L, M, N integrals within the framework of CL theory the field around the CL must be known. In order to evalu-

ate the stress field a mathematical model of the microstructure of the CL must be introduced. Such a model has been proposed in [32]. This model considers the micromechanics of the CL, i.e. the interaction of a macrocrack (or main crack) with an array of microdefects in a close vicinity of a macrocrack tip. The work also outlined the method for evaluation of a stress field around the CL. The present work develops the method of [32].

The model of work [32] treats a CL as a crack-cut in linear elastic medium surrounded by microcracks in the crack tip zone. The microcrack array surrounding the macrocrack tip consists of a field of randomly distributed small cracks (small in comparison to the macrocrack) of random lengths and orientations. The problem is to find the elastic stress field resulting from interaction of an array with the main crack subjected to external tractions. The main crack, with the microcrack array surrounding it, has been considered within the framework of the plane problem of elasticity. Also, the assumptions of small scale microcracking were supposed to hold [32]. This means that the microcrack array occupies a small area in comparison with the main crack tip. Under this assumption the macrocrack asymptotic stress field appears to be dominating and it is defined by the stress intensity factor K_1^0 only.

For the sake of simplicity, the mode I loading conditions have been assumed to hold. This explains the notation for intensity factor K_1^0, superscript "0" refers to the main crack.

The method of potentials has been chosen as a means of solution

of the problem. The reasons for that selection are: the possibility of generalization of the method for 3D problems, and the convenience of the method for statistical purposes.

The method of potentials gives the elastic stress field in a form of integrals of the potential density multiplied by Green's function and, therefore, the solution explicitly depends on the microcrack array configuration. This form of a stress field solution permits a relatively easy statistical averaging procedure.

2. Mathematical Formulation of the Problem

The macrocrack (or main crack) interacting with an array of microcracks under the assumptions of small scale microcracking (definition follows) is being considered. The two-dimensional linear elastic solid contains a macrocrack of length "$2\ell_0$" and an adjacent array of N rectilinear microcracks of lengths "$2\ell_i$" each, with $\underset{\sim}{n}_i$ as a unit normal vector to the i-th microcrack, and $\underset{\sim}{x}_i$ as a position vector on the i-th microcrack. i = 1, 2, ..., N. The elastic plane is under mode I tensile loading with respect to the main crack.

The stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ can be represented as a superposition

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{\sigma}_{\infty} + \hat{\underset{\sim}{\sigma}}(x) + \sum_{i=1}^{N} \underset{\sim}{\sigma}_i(\underset{\sim}{x}) \tag{1.1}$$

where $\underset{\sim}{\sigma}^{\infty}$ is the stress field due to remotely applied loads in the absence of cracks, $\hat{\underset{\sim}{\sigma}}$ and $\underset{\sim}{\sigma}_i(\underset{\sim}{x})$ are the stress fields generated by the main crack and by the i-th microcrack, respectively. More

exactly, σ_i is the stress field in an infinite solid containing one crack (references will be made to the microcrack ℓ_i which is just i-th microcrack) with faces loaded by tractions $\underset{\sim}{n}_i$ $[\sigma\infty + \hat{\underset{\sim}{\sigma}}(x_i) + \sum_{K=1, K\neq i}^{N} \underset{\sim}{\sigma}_K (X_i)]$ where $\hat{\underset{\sim}{\sigma}}(X_i)$ and $\underset{\sim}{\sigma}_K(\underset{\sim}{X}_i)$ are actual stresses generated by the main crack and the k-th microcrack along the line of ℓ_i.

In the vicinity of the microcrack tip, stresses $\underset{\sim}{\sigma}_\infty$ can be neglected compared to the tip-dominated field

$$\hat{\underset{\sim}{\sigma}}(\underset{\sim}{x}) = K_1^{eff} \frac{\phi[\theta(\underset{\sim}{x})]}{\sqrt{2\pi r(\underset{\sim}{x})}}$$

Where K_1^{eff} denotes the stress intensity factor for the macrocrack tip with the effect of microcracks taken into account, r and θ denote the position - vector and polar angle in the main crack tip corrdinate system. The small scale model is defined by the condition $\underset{\sim}{\sigma}_\infty << \underset{\sim}{\sigma}(\underset{\sim}{x})$. Thus, the asymtotic stress field in the vicinity of the main crack tip can be represented in the form.

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \frac{\phi[\theta(\underset{\sim}{x})]}{\sqrt{2\pi r(\underset{\sim}{x})}} K_1^{eff} + \sum_{i=1}^{N} \underset{\sim}{\sigma}_i(\underset{\sim}{x}) \qquad (1.2)$$

The technique of double layer potentials, with a potential density as a crack opening displacement (i.e. displacement discontinuity) will be used [33]. The displacement field may be represented by means of double layer potential density as follows:

$$\underset{\sim}{u}(\underset{\sim}{x}) = \int_{\ell} \underset{\sim}{b}(\xi) \cdot \underset{\sim}{\flat}(\underset{\sim}{\xi}, \underset{\sim}{x})d\xi \qquad (1.3)$$

where $\underset{\sim}{b}(\xi)$ is the crack opening displacement, and $\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x})$ is the second Green's tensor of elasticity for plane stress conditions. (Here and below through the whole work $d\xi$ must be understood as the increment along the crack.) The tensor $\underset{\sim}{\Phi}(\xi, x)$ constructed in Appendix I may be written as follows:

$$\underset{\sim}{\Phi}(\underset{\sim}{\xi},\underset{\sim}{x}) = \frac{1 + \nu}{4\pi R^2}\left[(1 - 2\nu)(\underset{\sim}{n}_x \underset{\sim}{R} - \underset{\sim}{R}\underset{\sim}{n} - \underset{\sim}{n}_x \underset{\sim}{R}\underset{\sim}{E}) - 2\frac{\underset{\sim}{n}_x \cdot \underset{\sim}{R}}{R^2}\underset{\sim}{R}\underset{\sim}{R}\right] \qquad (1.4)$$

where $\underset{\sim}{R} = \underset{\sim}{\xi} - \underset{\sim}{x}$, ν - is Poisson's ratio, E - is the second rank unit tensor, and the factor $1 + \nu$ is to be substituted by the factor of $1/(1 - \nu)$ for plane strain conditions.

Differentiating the displacement field (1.3) and taking the symmetrical part of a tensor gradient, the strain tensor can be obtained. Application of Hook's law to the strain tensor results in the stress tensor $\underset{\sim}{\sigma}(\underset{\sim}{x})$:

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{T}_x \int_{\ell} \underset{\sim}{b}(\underset{\sim}{\xi}) \cdot \underset{\sim}{\Phi}(\underset{\sim}{\xi},\underset{\sim}{x}) d\xi \qquad (1.5)$$

Where $\underset{\sim}{T}_x$ is the stress operator transforming the displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ into a stress field (subscript x indicates that differentiation in $\underset{\sim}{T}_x$ is performed with respect to x). Thus, the stress field (1.2) can be represented in the form

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \frac{\underset{\sim}{\phi}[\theta(\underset{\sim}{x})]}{\sqrt{2\pi r(\underset{\sim}{x})}} K_1^{eff} + \sum_{i=1}^{N} \underset{\sim}{T}_x \int_{\ell_i} \underset{\sim}{b}_i(\underset{\sim}{\xi}) \cdot \underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x}) d\underset{\sim}{\xi} \qquad (1.6)$$

where N unknown functions $\underset{\sim}{b}_i(\underset{\sim}{\xi})$ are to be determined from N vectorial integral equations expressing boundary conditions on the microcracks ℓ_i. The equilibrium equations are automatically satisfied for both terms in the superposition formula (1.6). The first term yields equilibrium equations because of the properties of the asymtotic crack solution, the second - because of the properties of the second Green's tensor $\underset{\sim}{\phi}(\underset{\sim}{\xi}, \underset{\sim}{x})$.

The faces of the microcracks must be traction free:

$$\underset{\sim}{n}_i\{\underset{\sim}{\sigma}(\underset{\sim}{x}_i) + \sum_{\substack{K=1\\K\neq i}}^{N} \underset{\sim}{T}_x \int_{\ell_K} \underset{\sim}{b}_K(\underset{\sim}{\xi})\cdot\underset{\sim}{\Phi}(\underset{\sim}{\xi},\underset{\sim}{x}_i)d\xi + \underset{\sim}{T}_x \int_{\ell_i} \underset{\sim}{b}_i(\underset{\sim}{\xi})\cdot\underset{\sim}{\Phi}(\underset{\sim}{\xi},\underset{\sim}{x}_i)d\xi\} = 0 \tag{1.7}$$

for all $x \in \ell_i$, for each i.

The last integral in the braces is to be understood in the principal value sense. It should be noted that the last integral in (1.7) becomes divergent if stress operator is moved under the integral sign and applied directly to the Green's tensor $\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x})$. It can be shown, however, that the limiting value of the integral is given by the following regularization:

$$\lim_{\underset{\sim}{x}\to\underset{\sim}{x}_i} \underset{\sim}{T}_x \int_{\ell} \underset{\sim}{b}(\underset{\sim}{\xi})\cdot\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x})d\xi = \int_{\ell} [\underset{\sim}{b}(\underset{\sim}{\xi}) - \underset{\sim}{b}(\underset{\sim}{x})]\cdot\underset{\sim}{T}_x\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x})d\xi \tag{1.8}$$

where the integral on the right must be understood in the principle value sense [34].

Expression (1.8) contains one more unknown - K_1^{eff}. An additional equation reflecting the impact of the microcrack array on the main

crack may be written in a form

$$K_1^{eff} = K_1^o + \frac{1}{\sqrt{\pi \ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + x}{\ell_o - x}}\, \underset{\sim}{n}(\underset{\sim}{x}) \sum_{i=1}^{N} \underset{\sim}{\sigma}_i(\underset{\sim}{x}) \cdot \underset{\sim}{n}(x) dx \qquad (1.9)$$

in the small scale microcracking model.

The last equation is an exact substitution for the boundary condition on the main crack.

Thus, the system of 2N + 1 scalar equations (1.7,1.9) represents the formulation of the main crack-microcrack array interaction problem. In the following the system of equations (1,7,1.9) is solved for two and three crack interaction problems.

In Chapter II, three particular problems are solved under the assumption of macrocrack dominating stress field to be piecewise constant on each microcrack. This is an approximate solution of the basic system of equations which can be justified for ratios $\ell_i/\ell_o <<$ 1 i = 1, 2, ... N).

In Chapter III, the higher order approximations are considered for two crack interaction problems. It is shown that the method can be extended to higher approximations and become exact in the limiting case.

In the final Chapter IV, the expression for stress field in a general problem of interaction of a macrocrack with the microcrack array of arbitrary configuration is obtained. It is shown that the resulting stress field can be fully characterized by asmytotic stress field of the macrocrack $\underset{\sim}{\sigma}_0(\underset{\sim}{x})$, (in absence of the microcrack

array) the values of its derivatives in the directions of microcracks evaluated at the centers of microcracks, and the second Green's tensor $\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x})$ constructed in the appendix I.

The obtained elastic stress field solution has been used for evaluation of the J-integral for the CL.

CHAPTER II

Piecewise Constant Approximation in Two and Three Crack Interaction Problems

1. Two Crack Interaction Problem (two colinear cracks)

In this section, the problem of elastic interaction of a macrocrack with one microcrack located on the same line is considered. The problem is solved based on the formulation of the previous chapter (i.e. plane stress, small scale model, mode I loading conditions) when the dominating stress field $\underset{\sim}{\hat{\sigma}}(x)$ on the microcrack ℓ is approximated by a constant.

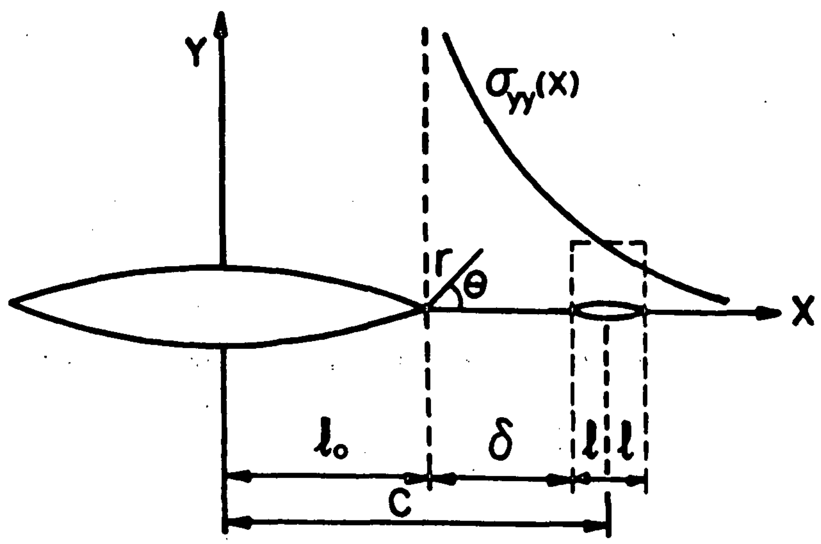

Figure 5

Under mode I external loading, the dominating stress field of the main crack of length $2\ell_0$ is given by

$$\hat{\sigma}(x) = K_1^{eff} \frac{\Phi[\theta(x)]}{\sqrt{2\pi r(x)}} \tag{2.1}$$

The stress field due to the microcrack ℓ is given by (1.6):

$$\sigma^{\ell}(x) = T_x \int_{c-\ell}^{c+\ell} b(\xi)\cdot\Phi(\xi, x)d\xi \tag{1.6}$$

where $b(\xi)$ is the microcrack opening displacement, (i.e., double layer potential density on ℓ). In the case of a constant approximation, microcrack ℓ is found to be embedded into a uniform stress field. The COD $b(\xi)$ in a uniform stress field is known to be elliptic [35] and may be written for mode I conditions:

$$b(\xi) = \frac{4\ell}{E} n(c)\cdot\sigma(c)\, e(\xi) \tag{2.2}$$

where E-is Young's modulus, $n(c)$-is unit normal vector to the crack ℓ at its center, $\sigma(c)$-is the resulting stress tensor at the microcrack center, and $e(\xi) = 1 - \frac{(\xi-\ell)^2}{\ell^2}$ is the elliptic crack opening ($n(c)\cdot\sigma(c)$ is a traction vector at point c).

The boundary conditions (1.7), in view of (2.2), gives

$$\sigma_{22}^{\ell}(x) = \hat{\sigma}_{22}(x) = K_1^{eff} \frac{1}{\sqrt{2\pi(\ell+\delta)}},\ x\in(c-\ell,\ c+\ell) \tag{2.3}$$

where superscript "ℓ" refers to the microcrack. The equation (2.3) takes scalar form because of the symmetry of the system of cracks. The stress component $\sigma_{22}^{\ell}(x)$ appears to be proportional to K_1^{eff} on the microcrack (note that $\sigma_{22}^{\ell}(x) = \sigma_{22}(c)$ when x (c - ℓ, c + ℓ)).

Therefore, $b(\xi)$, given by (2.2), also becomes proportional to K_1^{eff}. Thus, the last equation (1.9) for determination of effective stress intensity factor becomes linear algebraic equation with respect to K_1^{eff}. Equation (1.9) may be written as follows:

$$K_1^{eff} = K_1^o + \frac{1}{\sqrt{\pi\ell_o}} \int_{\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + x}{\ell_o - x}}\, \sigma_{22}^{\ell}(x)dx \qquad (2.4)$$

In the last equation, $\sigma_{22}^{\ell}(x)$ must be calculated from (1.6) with (2.2) as a double layer potential density.

The displacement vector $\underset{\sim}{u}(\underset{\sim}{x})$ is given by

$$\underset{\sim}{u}(\underset{\sim}{x}) = \frac{4\ell}{E} \int_{\ell} e(\xi)\underset{\sim}{n}(\xi)\cdot\underset{\sim}{\sigma}(\xi)\cdot\underset{\sim}{\Phi}(\xi, \underset{\sim}{x})d\xi \qquad (2.5)$$

where the product of tensors in the integrand may be written in index notation as $n_\alpha(\xi).\sigma_{\alpha\beta}(\xi).\Phi_{\beta\gamma}(\xi.\underset{\sim}{x})$ with $\alpha, \beta, \gamma = 1,2$ (summation by the repeated subscripts is implied). In the coordinate system of Figure 5 the integrand becomes: $e(\xi).\ (\sigma_{21}\Phi_{1\alpha} + \sigma_{22}\Phi_{2\alpha})$.
For evaluation of the stress component $\sigma_{22}^{\ell}(x)$ only the derivatives $\frac{\partial u_1}{\partial x_1}$ and $\frac{\partial u_2}{\partial x_1}$ need to be calculated due to the Hook's law:

$$\sigma_{22} = \mu\left[\frac{\nu}{1 - 2\nu}\left(\frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2}\right) + \frac{\partial u_2}{\partial X_2}\right] \qquad (2.6)$$

$$\frac{\partial u_1}{\partial X_1} = \frac{4\ell}{E} \int_{c-\ell}^{c+\ell} e(\xi) \quad (\sigma_{21} \frac{\partial \Phi_{11}}{\partial X_1} + \sigma_{22} \frac{\partial \Phi_{21}}{\partial X_2}) d\xi$$

$$\frac{\partial u_2}{\partial X_2} = \frac{4\ell}{E} \int_{c-\ell}^{c+\ell} e(\xi) \quad (\sigma_{21} \frac{\partial \Phi_{12}}{\partial X_2} + \sigma_{22} \frac{\partial \Phi_{22}}{\partial X_2}) d\xi \tag{2.7}$$

where

$$\Phi_{11}(\xi,\underset{\sim}{X}) = - \frac{1+\nu}{4\pi R^2} [(1-2\nu)X_2 + 2\frac{X_2}{R^2}(\xi_1 - X_1)^2]$$

$$\Phi_{21}(\xi,\underset{\sim}{X}) = \frac{1+\nu}{4\pi R^2}[(1-2\nu)(\xi_1 - X_1) - 2\frac{X_2^2}{R^2}(\xi_1 - X_1)]$$

$$\Phi_{12}(\xi,\underset{\sim}{X}) = - \frac{1+\nu}{4\pi R^2}[(1-2\nu)(\xi_1 - X_1) - 2\frac{X_2^2}{R^2}(\xi_1 - X_1)] \tag{2.8}$$

$$\Phi_{22}(\xi,\underset{\sim}{X}) = - \frac{1+\nu}{4\pi R^2}[(1-2\nu)X_2 + 2\frac{X_2^2}{R^2}]$$

in the chosen coordinate system, with $R^2 = (\xi_1 - x_1)^2 + x_2^2$ (the subscripts "1" and "2" refer to the x and y axis, respectively).

Substitution of (2.8) and (2.7) into (2.6), results in

$$\sigma_{22}^{\ell}(X) = \frac{\sigma_{22}(c)}{\pi} \int_{c-\ell}^{c+\ell} \frac{\sqrt{-\xi^2 + 2c\xi - (c^2 - \ell^2)}}{(\xi - X)^2} d\xi \tag{2.9}$$

where x and ξ are coordinates on the horizontal axis.

Integration gives: (The integral in (2.9) is evaluated in Appendix II)

$$\sigma_{22}(X) = \sigma_{22}(C) \left(\frac{1}{\sqrt{1 - (\frac{\ell}{X-c})^2}} - 1\right) \tag{2.10}$$

where $x - c \geq \ell$.

Substituting (2.10) into (2.4) we obtain K_1^{eff} as follows:

$$K_1^{eff} = K_1^o + \frac{\sigma_{22}(C)}{\sqrt{\pi\ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \left(\frac{1}{\sqrt{1 - (\frac{\ell}{X-c})^2}} - 1\right) dX \tag{2.4a}$$

The second term in (2.4a) represents an increment ΔK of stress intensity factor due to the presence of the microcrack.

$$\Delta K = \frac{\sigma_{22}(C)}{\sqrt{\pi\ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \left(\frac{1}{\sqrt{1 - (\frac{\ell}{X-c})^2}} - 1\right) dX \tag{2.11}$$

The term decreases with the increase of c (when $|x - c| \gg \ell$) and tends to infinity when $c \to \ell_o + \ell$, i.e., when $\delta \to 0$ (the distance between the adjacent crack tip tends to zero).

The combination of (2.3) $\left(\sigma_{22}(c) = K_1^{eff} \frac{1}{\sqrt{2\pi(\ell + \delta)}}\right)$ and (2.4a) results in linear algebraic equation for K_1^{eff}:

$$K_1^{eff} = K_1^o + K_1^{eff} q(\delta/\ell) \tag{2.12}$$

where

$$q(\delta/\ell) = \frac{1}{\sqrt{\ell' + \delta'}} \int_{-1}^{1} \sqrt{\frac{1 + X}{1 - X}} \frac{1}{\sqrt{1 - (\frac{\ell}{X-c})^2}} - 1) dX \tag{2.13}$$

and $\ell' = \ell/\ell_o$, $\delta' = \delta/\ell_o$, $C' = C/_o$ are nondimensionalized coordinates with respect to the main crack length ℓ_o. From dimensional

considerations it follows that $q = q(\delta/\ell)$ depends on ratio of the only two characteristic lengths of the problem (δ and ℓ).

The solution of (2.12) is obvious

$$K_1^{eff} = \frac{K_1^o}{1 - q} \tag{2.14}$$

The graph of K_1^{eff}/K_1^0 is given in Figures 6 & 7. The graph in Figure 7 is presented in order to illustrate the behavior of K_1^{eff}/K_1^o for small δ/ℓ.

The obtained result indicates that effective stress intensity factor K_1^{eff} increases from K_1^0 to infinity when the distance between two cracks tends to zero.

General superposition formula for stresses (1.2) can be written now as follows: $\underset{\sim}{\sigma}(\underset{\sim}{X}) = \underset{\sim}{\theta}(\underset{\sim}{X}) + \underset{\sim}{\sigma}^{\ell}(\underset{\sim}{X}) =$

$$K_1^{eff} \left\{ \frac{\phi[\theta(X)]}{\sqrt{2\pi r(\underset{\sim}{X})}} + \frac{1}{\sqrt{2\pi(\ell + \delta)}} \frac{4\ell}{E} \underset{\sim}{n}(C)\underset{\sim X}{T} \int_{c-\ell}^{c+\ell} e(\xi)\underset{\sim}{\Phi}(\xi,\underset{\sim}{X})d\xi \right. \tag{2.15}$$

Formula (2.15) with K_1^{eff} given by (2.14) represents the approximate solution of our problem (piecewise-constant approximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$.

The first term in the solution (2.15) ($K_1^{eff} \frac{\underset{\sim}{\phi}[\theta(x)]}{\sqrt{2\pi r(\underset{\sim}{x})}}$) represents the dominating stress field of the main crack. The second term

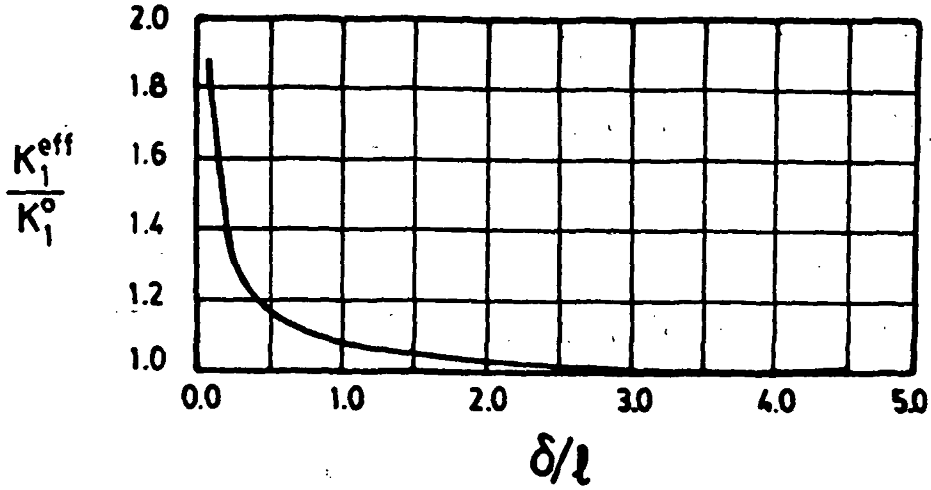

Figure 6

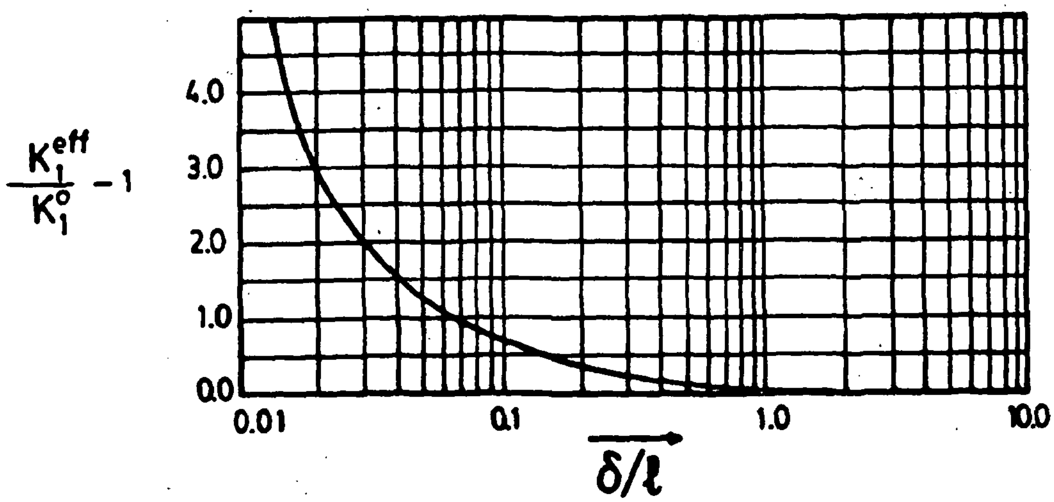

Figure 7

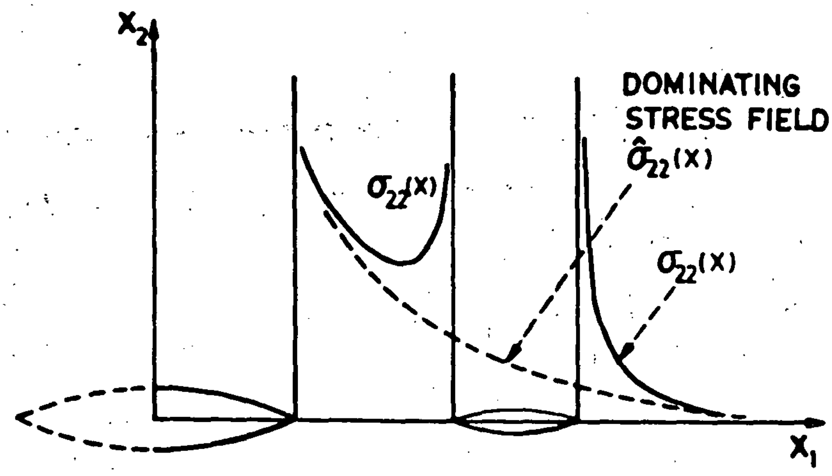

Figure 8

represents the stress field of the microcrack with the magnitude of microcrack opening displacement determined by the magnitude of dominating stress field at the center of the microcrack. Figure 8 gives qualitative graph of the stress distribution following from formula (2.15).

One more comment should be made with respect to the solution of the system of equations (2.3), (2.4). This system has been reduced to single equation (2.12) for K_1^{eff}. (Of course, similar equations can be written for $\sigma_{22}(c)$). In the problem under consideration, i.e., problem of interaction of two cracks, the obvious solution of (2.12) is given by (2.14). However, in more complicated situations, i.e. in many cracks interaction problems, it may be useful to try an approximate methods for solution of a system of equations corresponding to (2.3), (2.4). It is easy to see the meaning of certain approximations in the case of simple problem under consideration.

The [32] work suggested an iterative procedure as an alternative

to an exact solution. In our problem, equation (2.12) can be solved by means of an iterative process. If one takes K_1^0 as a zero approximation for K_1^{eff} in (2.12), then an iterative process gives geometric series

$$K_1^{eff} = K_1^o(1 + q + q^2 + q^3 + \ldots) \tag{2.16}$$

and the sum of this series for $|q| < 1$ coincides with (2.14). On the other hand, formula (2.14) is meaningful only for $|q| < 1$, therefore (2.14) and (2.16) are equally valid.

The sequence of iterated terms has clear physical meaning: the first term gives the intensity factor of a main crack k_1^o in the absence of a microcrack, the second term accounts for first order interaction, i.e.. microcrack, being imbedded in the field of main crack K_1^o, gives the correction to K_1^0 of a magnitude $k_1^o q$. The third term accounts for double interactions, etc.

Substituting (2.16) into (2.15) we obtain the formula each term of which can be interpreted by means of diagram in Figure 9. The first term of the sum represents the stress field of the main crack under the loading σ_∞ in the absence of the microcrack, the second term represents the stress field of the microcrack embedded into the main crack field, the third term represents the correction to the main crack stress field resulting from presence of the microcrack which is embedded into the stress field of the main crack, the fourth term represents the correction to the microcrack stress field resulting from the correction to the main crack field, etc. The solution

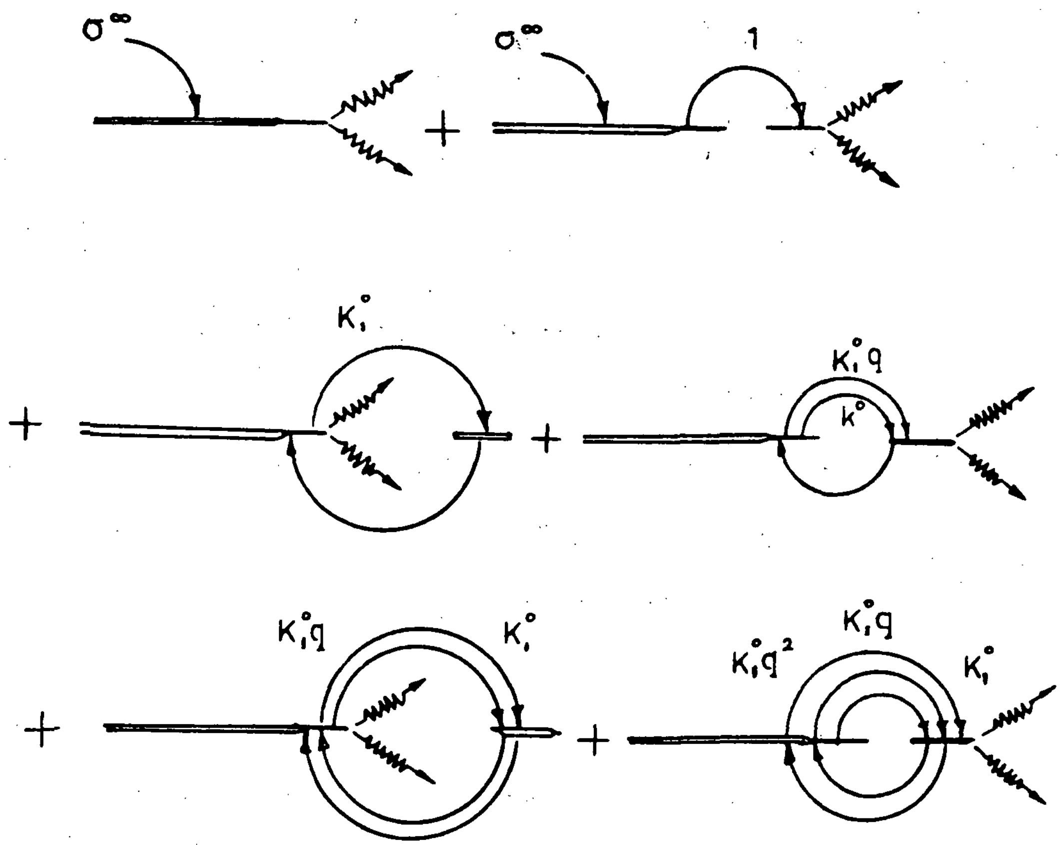

Figure 9

(2.15) with K_1^{eff} given by (2.16) gives an approximation in two different senses: the approximation of a stress field $\sigma(x)$ by a constant on the microcrack, and an approximation due to a number of physical interactions between the cracks taken into account.

2. Two Crack Interaction Problem (two parallel cracks)

In this section the problem of elastic interaction of two parallel cracks is considered, (see Figure 10) i.e., macrocrack of a length "$2\ell_0$" and a microcrack of length "2ℓ".

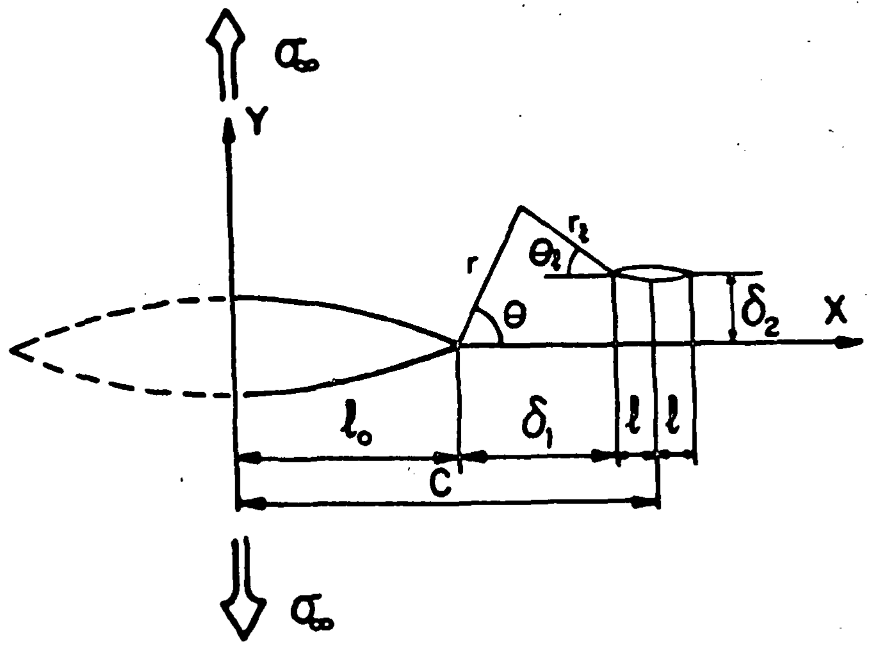

Figure 10

Constant approximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ on the microcrack has been used again. For simplicity purposes instead of a double layer potentials technique we use the asymptotic stress field solution for the microcrack.

The purpose of this section is to show that the method works for the problem which has not been solved so far (to our knowledge). The solution has been obtained under the simplest possible assumptions as an illustration of the method.

The system of cracks in Figure 10 is assumed to be under mode I tensile loading. The small scale microcracking assumption also holds. The dominating stress field is given by (2.1). The σ_{22}^{ℓ} stress component of the microcrack field in $r_\ell(x)$, $\theta_\ell(x)$ coordinate system may be written as follows:

$$\sigma_{22}^{\ell}(X) = K_1 \frac{\phi_{22}[\theta_\ell(\underset{\sim}{X})]}{\sqrt{2\pi r_\ell(\underset{\sim}{X})}} + K_2 \frac{\phi_{21}[\theta_\ell(\underset{\sim}{X})}{\sqrt{2\pi r_\ell(\underset{\sim}{X})}} \tag{2.17}$$

where the first term represents the contribution of the mode I asymptotic stress field, and the second term accounts for the mode II. (The resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ in this problem contains both modes of loading, of course.) The stress intensity factors K_1 and K_2 are given by

$$K_1 = \sigma_{22}(C,\delta_2)\sqrt{\pi\ell}\ , \ K_2 = \sigma_{21}(C,\delta_2)\sqrt{\pi\ell} \tag{2.18}$$

and

$$\phi_{22}(\theta_\ell) = \cos\frac{\theta_\ell}{2}\left(1 + \sin\frac{\theta_\ell}{2}\sin\frac{3\theta_\ell}{2}\right),\ \phi_{21}(\theta_\ell) = \cos\frac{\theta_\ell}{2}\sin\frac{\theta_\ell}{2}\sin\frac{3\theta_\ell}{2}$$

The resulting stress field $\underset{\sim}{\sigma}(x)$ acts as an external field applied to the microcrack and under the constant stress approximation, equations (2.18) hold.

The boundary conditions on the microcrack (1.7) take form

$$\begin{aligned} \sigma_{22}(C,\delta_2) &= K_1^{eff} \frac{\phi_{22}(\theta_o)}{\sqrt{2\pi r_o}} \\ \sigma_{21}(C,\delta_2) &= K_1^{eff} \frac{\phi_{21}(\theta_o)}{\sqrt{2\pi r_o}} \end{aligned} \tag{2.19}$$

where $\theta_o = \tan^{-1}\frac{\delta_2}{\delta_1+\ell}$ and $r_o^2 = (\delta_1 + \ell)^2 + \delta_2^2$ are coordinates of the center of the microcrack in the main crack tip coordinate

system. The equation (1.9) for effective stress intensity factor appears as follows:

$$K_1^{eff} = K_1^o + \sigma_{22}(C,\delta_2)\sqrt{\frac{\ell}{\ell_o}}\, f_2 + \sigma_{21}(C,\delta_2)\sqrt{\frac{\ell}{\ell_o}}\, f_1 \qquad (2.20)$$

where

$$f_1 = \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \cdot \frac{\phi_{21}[\theta_\ell(X)]}{\sqrt{2\pi r_\ell(X)}}\, dx \qquad (2.21)$$

$$f_2 = \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \cdot \frac{\phi_{22}[\theta_\ell(X)]}{\sqrt{2\pi r_\ell(X)}}\, dx$$

where relations $\theta_\ell = \theta_\ell(x)$ and $r_\ell = r_\ell(x)$ are given by (see Figure 10). $r_\ell(X) = (\ell_o + \delta_1 - X)^2 + \delta_2^2$, and $\theta_\ell(X) = \sin \frac{\delta_\ell}{r_\ell(X)}$

The factors $\sigma_{22}(C,\delta_2)\ \frac{\phi_{22}(\theta_\ell)}{\sqrt{2\pi r_\ell}}$ and $\sigma_{21}(C,\delta_2) = \frac{\phi_{21}(\theta_\ell)}{\sqrt{2\pi r_\ell}}$ in (2.20) characterize the $\sigma_{22}^\ell(x)$ component of stress on the main crack.

Weight functions f_1 and f_2 depend on the microcrack location and orientation. The dependence of K_1^{eff} upon the length of the microcrack is given by the factor of $\sqrt{\frac{\ell}{\ell_o}}$. Equations (2.19), (2.20) constitute a system of three linear algebraic equations with three unknowns : $\sigma_{22}(C, \delta_2)$, $\sigma_{21}(C, \delta_2)$ and K_1^{eff}. Its solution yields formula (2.14), where

$$q(\delta_1/\ell,\ \delta_2/\ell) = \sqrt{\frac{\ell}{\ell_o}} \cdot \frac{1}{\sqrt{2\pi r_o}}\ [f_2\ \phi_{22}(\theta_o) + f_1\ \phi_{21}(\theta_o) \qquad (2.22)$$

and stress components are given by (2.19).

General superposition formula (1.2) with the help of (2.18) and (2.19), gives the asymptotic stress field in the vicinity of the microcrack in the form:

$$\underset{\sim}{\sigma}(\underset{\sim}{X}) = K_1^{eff}\ \{ \frac{\underset{\sim}{\phi}[\theta(\underset{\sim}{X})]}{\sqrt{2\pi r(\underset{\sim}{X})}} + \sqrt{\frac{\ell}{2r_o}} \cdot [\phi_{22}(\theta_o)\ \frac{\underset{\sim}{\phi}[\theta_\ell(\underset{\sim}{X})]}{\sqrt{2\pi r_\ell(\underset{\sim}{X})}} + \phi_{21}(\theta_o)\ \frac{\underset{\sim}{\phi}[\theta_\ell(\underset{\sim}{X})]}{\sqrt{2\pi r_\ell(\underset{\sim}{X})}}]\ \} \qquad (2.23)$$

The presence of mode II loading in the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ gives rise to K_{11}^{eff} in this problem. The shear mode contribution is represented by the last terms in formulas (2.22) and (2.23).

In order to make explicit the dependence of K_1^{eff} on the parameters of a crack system (Figure 10) the function $q(\delta_1/\ell,\ \delta_2/\ell)$ must be evaluated. This function, in fact, depends on ratios of δ_1/ℓ, δ_2/ℓ only, as follows from dimensional analysis.

Let us consider the expression.

$$\frac{f_2}{\sqrt{2\pi r_o}} = \frac{1}{\sqrt{2\pi r_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \cdot \frac{\phi_{22}[\theta_\ell(X)]}{\sqrt{2\pi r_\ell(X)}}\ dx \qquad (2.24)$$

The integrand in (2.24) has a singularity at $x = \ell_o$ (at the right main crack tip) of a square root type. The factor at $\sqrt{\frac{\ell_o + X}{\ell_o - X}}$ is restricted by the inequality $0 \le \frac{\phi_{22}[\theta_\ell(X)]}{\sqrt{2\pi r_\ell(X)}} \le \frac{1.3}{\sqrt{2\pi r_o}}$ which may be narrowed for $|\theta_\ell| < \pi$ (this is our case because the interaction of only left microcrack tip with the main crack has been taken into

account). Thus, by the mean value theorem (2.24) may be represented as follows:

$$\frac{f_2}{\sqrt{2\pi r_o}} = \frac{\alpha}{2\pi r_o} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}}\, dx = \frac{\alpha \ell_o}{2r_o} \tag{2.25}$$

(note that $\int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}}\, dx = \ell_o \pi$), where $0 \leq \alpha \leq 1.3$. The same line of reasoning is applicable to the second term $q(\delta_1/\ell\ ,\ \delta_2/\ell)$ in formula (2.22),

$$\frac{f_1}{\sqrt{2\pi r_o}} = \frac{1}{\sqrt{2\pi r_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \cdot \frac{\phi_{21}[\theta_\ell(X)]}{\sqrt{2\pi r_\ell(X)}}\, dx = \frac{\beta \ell_o}{2r_o} \tag{2.26}$$

where $-0.4 \leq \beta \leq 0.2$. (interval for β can be narrowed also). Using (2.25), (2.26), and (2.22) $q = q(\delta_1/\ell\ ,\ \delta_2/\ell)$ takes the form

$$q(\delta_1/\ell,\ \delta_2/\ell) = \sqrt{\frac{2\ell_o \ell}{2}} \cdot [\alpha\phi_{22}(\theta_o) + \beta\phi_{21}(\theta_o)] \tag{2.27}$$

Introducing $r_o = r_o(\ell,\ \delta_1,\ \delta_2)$ into the last expression and dividing both numerator and denominator by ℓ_o results in

$$q(\delta_1/\ell,\ \delta_2/\ell) = \frac{1}{2}\, [\alpha\phi_{22}(\theta_o) + \beta\phi_{21}(\theta_o)] \quad \frac{\ell'}{(\delta_1' + \ell')^2 + \delta_2'^2} \tag{2.27a}$$

where $\ell' = \ell/\ell_o$, $\delta_1' = \delta_1/\ell_o$, and $\delta_2' = \delta_2/\ell_o$ are nondimensionalized parameters.

Direct calculations show that the factor of $\frac{1}{2}[\alpha\phi_{22}(\theta_0) + \beta\phi_{21}(\theta_0)]$ is positive for any location of the microcrack, which means that effective stress intensity factor K_1^{eff} always increases because of the presence of the microcrack.

In the case of both, δ_1, $\delta_2 \to 0$ formula (2.22) for q may be shown to reduce to the corresponding formula of the previous section. Thus, the effective stress intensity factor tends to infinity when the distance between micro and macrocrack vanishes.

Qualitatively the behavior of effective stress intensity factor in this problem is described by (2.14) with q given by (2.27) or (2.27a).

3. Three Crack Interaction Problem

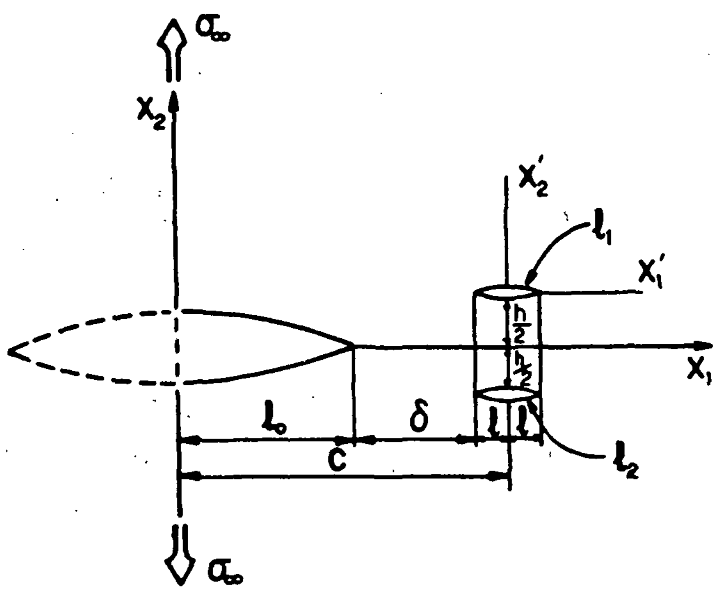

Figure 11

The purpose of this section is to analyze the problem of interaction of a macrocrack with two symmetrically located microcracks, Figure 11, under the assumptions of the previous two sections. Generalization in both directions nonparallel system of cracks and nonsymmetrical location of microcracks is possible but calculations become considerably more complicated.

Two distinct cases may occur, one when the microcrack array amplifies the dominating field (and this has happened in two previous problems), the other when the microcrack array diminishes the main crack field. The latter is called the "shielding" effect. The simplest situation for "shielding" to occur is when two parallel microcracks are placed at a distance $c = \ell_0$ (see Figure 11). The goal of this section is to estimate the effect of K_1^{eff} reduction when microcracks have been placed at the indicated position, and to show that in case of $c \gg \ell_0$ and h reasonably small two parallel microcracks act in a way similar to one microcrack in section 1.

There is no shear mode in the resulting stress field because of the symmetry of the problem. That is why the system of equations (1.7) (boundary conditions on microcracks) the form of two scalar equations

$$\sigma_{22}(C, \tfrac{h}{2}) + F_{21}\,\sigma_{22}(C, -\tfrac{h}{2}) = -K_1^{eff}\,\frac{\phi_{22}[\theta(C, \frac{h}{2})]}{\sqrt{2\pi r(C, \frac{h}{2})}}$$

$$\sigma_{22}(C, -\tfrac{h}{2}) + F_{12}\,\sigma_{22}(C, \tfrac{h}{2}) = -K_1^{eff}\,\frac{\phi_{22}[\theta(C, -\frac{h}{2})]}{\sqrt{2\pi r(C, -\frac{h}{2})}} \tag{2.28}$$

The first term in the first equation (2.28) represents the normal stress component at the center of microcrack ℓ_1, the second term represents the stress component σ_{22} which microcrack ℓ_2 exerts on ℓ_1, with F_{21} as a scalar influence function, and finally, the right hand term represents the dominating stress field at the center of microcrack ℓ_1. The equation is formulated at the center of microcrack ℓ_1 because of piecewise constant approximation assumption. The second equation has been formulated for the microcrack ℓ_2, and because of symmetry it is identical with the first one ($\sigma_{22}(c, \frac{h}{2}) = \sigma_{22}(c, -\frac{h}{2})$). The influence function $F_{12} = F_{21} = F$ (because of symmetry) appears from (1.6) with (2.2) as a double layer potential density (because the unknown field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ is assumed to be constant on a microcrack). It is defined by the expression:

$$F(\ell, \underset{\sim}{X})\underset{\sim}{\sigma} = \frac{4}{E} \; \underset{\sim}{n}(\underset{\sim}{X}) \; \underset{\sim}{n}(\underset{\sim}{X})\underset{\sim}{T}_x \int_{-\ell}^{\ell} e(\xi) \; \underset{\sim}{n}(\xi)\cdot\underset{\sim}{\sigma}(\xi)\cdot\underset{\sim}{\phi}(\xi, \underset{\sim}{X})d\xi \qquad (2.29)$$

where $\ell_1 = \ell_2 = \ell$, and $n(x)$ is a unit normal vector at point x in the direction of x_2 axis.

The expession above represents the influence function of a crack at any point $\underset{\sim}{x}$. The function $F(\ell_1, \underset{\sim}{x}')$ in a coordinate system of the center of the microcack takes the form

$$F(\ell, X') = \ell \int_{-\ell}^{\ell} e(\xi) \; \frac{3X_2'^4 - (\xi - X_1')^4 - (\xi - X_1')^2 X_1'^2}{[(\xi - X_1')^2 + X_2'^2]^3} \, d\xi \qquad (2.30)$$

where position vector $\underset{\sim}{x}'$ has the components x_1' and x_2'. (The integra-

tion procedure is described in Appendix II).

In addition to the system of equations (2.28), there is an equation for the effective stress intensity factor K_1^{eff} (1.9). For this problem it can be written in the form

$$K_1^{eff} = K_1^o + \frac{2}{\sqrt{\pi \ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}}\, \sigma_{22}^{\ell}(X)dx \tag{2.31}$$

where the factor of 2 in the second term appears due to the presence of two symmetrically located microcracks, and $\sigma_{22}(x)$ with help of (1.6) and (2.2) may be represented as

$$\sigma_{22}^{\ell}(X) = \frac{4}{E}\, n(X)\, n(X)\, \sigma(C, \tfrac{h}{2})\, \underset{\sim}{n}(C, \tfrac{h}{2})\, \underset{\sim}{T}_x \int_{\ell} e(\xi)\underset{\sim}{\phi}(\xi, \underset{\sim}{X})d\xi \tag{2.32}$$

where $\underset{\sim}{\sigma}(c, \frac{h}{2}) = \underset{\sim}{\sigma}(c, -\frac{h}{2})$, and $\underset{\sim}{n}(c, -\frac{h}{2}) = \underset{\sim}{n}(c, -\frac{h}{2})$, and because of that (2.32) holds for both microcracks. Thus, three equations (2.28), (2.31), in view of (2.32), represent a system of linear algebraic equations for determination of three unknowns $\sigma_{22}(c, \frac{h}{2})$, $\sigma_{22}(c, -\frac{h}{2})$ and K_1^{eff}. (Because of symmetry, in fact, there are only two equations for two unknowns $\sigma_{22}(c, \frac{h}{2})$, and K_1^{eff}.

Substitution of $\sigma_{22}(c, \frac{h}{2}) = - \dfrac{K_1^{eff}}{V + F(\ell,0,h)} \dfrac{\phi_{22}[\theta(C,\frac{h}{2})]}{\sqrt{2\pi r(C,\frac{h}{2})}}$ from (2.28)

into (2.32) with the subsequent substitution of the latter into (2.31), results in the equation for K_1^{eff} with the solution in the usual form (2.14) with,

$$q = -\frac{2}{\pi \ell_o [1 + F(\ell, 0, h)]} \cdot \frac{\Phi_{22}[\theta(C, \frac{h}{2})]}{\sqrt{2\pi r(C, \frac{h}{2})}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}}\, F(\ell, \underset{\sim}{X})\, dx \qquad (2.33)$$

Influence function $F(\ell, \underset{\sim}{x})$ in (2.33) is defined by (2.29) and must be evaluated in the coordinate system of the main crack (Figure 11). This function may be obtained from (2.30) also by coordinate transformation $x_1' = x_1' - c$, $x_2' = x_2 - \frac{h}{2}$ and integrating from $(c - \ell)$ to $(c + \ell)$.

In order to analyze the behavior of the effective stress intensity factor K_1^{eff} the influence function $F(\ell, \underset{\sim}{x})$ must be evaluated. The integral in formula (2.30) can be evaluated exactly but it is enough for our puposes to estimate it with help of the mean value theorem: (Formula (2.34) is obtained from (2.30) in Appendix II).

$$f(\ell, \underset{\sim}{X}') = -\, e_o \frac{\ell}{X_2'} \{ 3\left(\frac{(X_1' - \ell)/X_2'}{1 + (X_1' - \ell)^2/X_2'^2} - \frac{(X_1' + \ell)/X_2'}{1 + (X_1' + \ell)^2/X_2'^2}\right) + \left(\frac{(X_1' - \ell)^3/X_2'^3}{[1 + (X_1' - \ell)^2/X_2'^2]^2} - \frac{(X_1' + \ell)^3/X_2'^3}{[1 + (X_1' + \ell)^2/X_2'^2]^2}\right) \} \qquad (2.34)$$

where $e_o = \sqrt{1 - \left(\frac{X_o - C}{\ell}\right)^2}$ is the elliptic crack opening at some point $x_o \in (c - \ell, c + \ell)$. The influence function $F(\ell, x')$ given by (2.34) can be considerably simplified for both large and small X_2':

$$F(\ell, \underset{\sim}{X}') \approx 6 \frac{\ell}{X_2'} \qquad (2.35)$$

for large X_2', i.e. $(X_1' - \ell)/X_2' << 1$ and $(X_1' + \ell)/X_2' << 1$.

$$F(\ell, \underset{\sim}{X}') \simeq -\ell\left(\frac{1}{X_1' - \ell} - \frac{1}{X_1' + \ell}\right) \qquad (2.36)$$

for small X_2', i.e. $(X_1 - \ell)/X_2' >> 1$ and $(X_1 + \ell)/x_2' >> 1$. In formulas (2.35) and (2.36) it was assumed that $e_0 = 1$. Substituting formulas (2.35) and (2.36) into (2.33) the estimates of K_1^{eff}/K_1^0 for small and large ℓ/h can be obtained. It follows from (2.35) that

$$F(\ell, 0, h) \simeq 6\,\frac{\ell}{h} \qquad (2.37)$$

for the microcracks being wide apart from the macrocrack tip, i.e. $\ell << h$.

Substituting (2.37) into (2.33), and taking into account that the second of (2.37) holds only for $\ell_0 - h < X_1 \leq \ell_0$, we obtain

$$q \doteq -\frac{3}{\pi} \cdot \frac{6}{1 + 6(\ell/h)^2} \cdot \left(\frac{\ell}{h/2}\right)^2 \quad \text{for small } \ell/h \qquad (2.38)$$

The last expession gives the uppe estimate of $q=q(\ell/h)$ because of the conditions e=1, and $\ell_0 - h < X_1 \leq \ell_0$.

Analogously, for the microcracks being close to the macrocrack tip, i.e. $\ell >> h$, the formula (2.36) gives

$$F(\ell, 0, h) \to 2 \quad \text{for small } h \qquad (2.39)$$

(The last expression represents the limiting value of the influence function (2.29) which is $2e_0$ for the approximation (2.34)).

Substituting (2.36) and (2.39) into (2.33) and evaluating the integral (see Appendix II) we obtain

$$q \approx \sqrt{\frac{2+\ell}{2} \cdot \frac{\ell}{h}} \qquad \text{for large } \ell/h \tag{2.40}$$

The important feature of (2.38) and (2.40) is that both of them give negative value for q which, in turn, gives the reduction of the

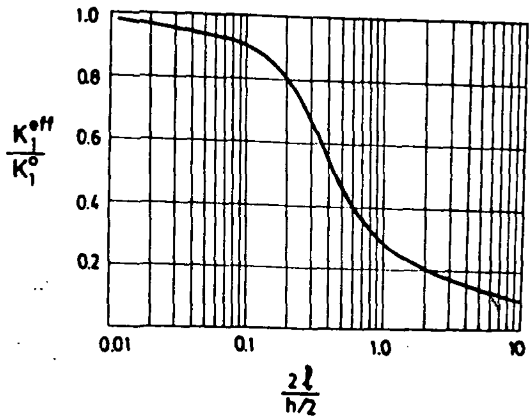

Figure 12

effective stress intensity factor K_1^{eff}. The graph of K_1^{eff}/K_1^{o} vs. $\frac{2\ell}{h/2}$ (i.e., the microcrack length over the vertical distance between the macro and microcrack, see figure (11)) is represented in figure (12).

The upper and the lower portions of the curve were calculated using (2.38) and (2.40), respectively. The effective stress intensity factor K_1^{eff} is always less than K_1^{o}. It varies from K_1^{o} for small 4 /h to zero for large 4 /h.

The effect of reduction of the effective stress intensity factor K_1^{eff} at small h is similar to the one obtained on Dagdale-Barenblatt model. The effect of vanishing of the stress intensity factor on

Dagdale-Barenblatt model has been obtained by means of introducing interaction forces between the crack faces in a crack tip zone. In our model the effect appears as a consequence of microcracking in a crack tip zone.

In the case when the system of microcracks is being moved far from the main crack tip in a horizontal direction, i.e., $c \gg \ell_0$, formula (2.30) gives negative value for $F(\ell, \underset{\sim}{x})$, and q in (2.33) becomes positive. Thus, the effective stress intensity factor K_1^{eff} increases in this case, just as in the case of one microcrack on the same line with the main crack. More similarity can be noticed by increasing the distance between the macro and microcrack centers at the given h. In this case the influence function $F(\ell, \underset{\sim}{x})$ in (2.33) may be reduced to the corresponding expression in the problem of Section I. In terms of function $F(\ell, \underset{\sim}{x}')$, formula (2.30) reduces to

$$F(\ell, \underset{\sim}{X}') \doteq \ell \int_{-\ell}^{\ell} e(\xi) \cdot \frac{d\xi}{(\xi - X_1')^2} ,$$

which coincides with the corresponding expression of Section I. Calculations show that $F(\ell, x_1')$ changes its sign from plus to minus (i.e., shielding vs. amplification of K_1^{eff} when $c < \ell_0 + \ell$ for $\ell/h = 1$ (see Figure 11).

CHAPTER III

Higher Order Approximations for the Problem of Two Colinear Cracks

1. Linear Approximation

In this chapter the problem of Chapter II, Section I, will be considered (Figure 5). All the assumptions of that section are assumed to hold, but instead of constant approximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ on the microcrack, linear approximation will be taken. Thus, the elastic stress field on the microcrack is assumed to be of a form

$$\underset{\sim}{\sigma}(X') = \underset{\sim}{\sigma}'(C')\ (X' - C') + \underset{\sim}{\sigma}(C'), \quad X' \in (C' - \ell', C' + \ell') \tag{3.1}$$

where $\underset{\sim}{\sigma}'(c')$ is the derivative with respect to x' at the center of the microcrack, $x' = x/\ell_o$, $c' = c/\ell_o$, etc. represent nondimensional coordinates.

According to the theorem on polynomial conservation (Willis's theorem) [35] a polynomial loading produces elliptic crack opening displacement multiplied by the polynomial of the same degree as the loading. Using this result the double layer potential density b(ξ) can be written in the form

$$\underset{\sim}{b}(\xi) = [b_o + b_1(\xi' - C')]\ 4e(\xi)\underset{\sim}{n}(\xi) \tag{3.2}$$

instead of (2.2) in the previous chapter. Unknown coefficients b_o and b_1 are of length units. By formula (1.3) the displacement vector may be represented as follows:

$$\underset{\sim}{u}(\underset{\sim}{X}) = \int_{C-\ell}^{C+\ell} [b_o + b_1(\xi - C)]\ 4\ e(\xi)\ \underset{\sim}{n}(\xi)\cdot\underset{\sim}{\Phi}(\xi, \underset{\sim}{X})d\xi$$

where $\quad n(\xi)\cdot\Phi(\xi, X) = n_\alpha(\xi)\Phi_{\alpha\beta}(\xi, X) = \Phi_{2\beta}(\xi, X)$

Application of the stress operator T_x to the displacement vector $\underset{\sim}{u}(\underset{\sim}{x})$ results in the stress field of the microcrack. The $\sigma_{22}(x)$ component of it may be written as follows:

$$\sigma_{22}^{\ell}(X') = \frac{E}{\pi\ell} \int_{C-\ell}^{C+\ell} \frac{e(\xi)}{(\xi' - X')^2} [b_o + b_1(\xi' - C')]'d\xi \qquad (3.3)$$

Formula (3.3) is valid on x_1 axis (i.e. $x_2 = 0$). In this chapter we will write x instead of x_1. Both coordinates x and ξ are on the horizontal axis. The integral in (3.3) diverges when $x \in (c - \ell, c + \ell)$ and must be understood in the sense described in Chapter I, Section 2. Evaluation of this integral gives,

$$\sigma_{22}^{\ell}(X') = \sigma_{22}'(C')(X' - C') + \sigma_{22}(C') = \frac{E}{\ell}[b_o + 2b_1(X' - C')] \quad (3.4)$$

and comparing coefficients of linear function we obtain the relations

$$\begin{aligned} \sigma_{22}(C') &= \frac{E}{\ell}b_o \\ \sigma_{22}'(C') &= 2\frac{E}{\ell}b_1 \end{aligned} \qquad (3.5)$$

Substitution of (3.5) into (3.3) expresses $\sigma^{\ell}_{22}(x)$ as a linear combination of $\sigma_{22}(c')$ and $\sigma'_{22}(c')$ as follows:

$$\sigma^{\ell}_{22}(X') = \sigma_{22}(C')\left[\frac{1}{\sqrt{1-\left(\frac{\ell}{X-C}\right)^2}} - 1\right] - \frac{1}{2}(C'-X')\sigma'_{22}(C')\left[\left(\frac{1}{\sqrt{1-\left(\frac{\ell}{X-C}\right)^2}} - 1\right) + \left(\sqrt{1-\left(\frac{\ell}{X-C}\right)^2} - 1\right)\right] \quad (3.6)$$

$$X \bar{\in} (C-\ell,\ C+\ell)$$

Equation (3.6) has been obtained from (3.3) by means of integration. (The integrals are evaluated in Appendix II.)

Following the procedure in section 1, Chapter II boundary condition (1.7) on the microcrack takes form (2.3) and, consequently

$$\sigma'_{22}(C) = -K_1^{eff}\,\frac{1}{\sqrt{2\pi(\ell+\delta)}}\cdot\frac{1}{2(\ell'+\delta')} \quad (3.7)$$

The equation for effective stress intensity factor (1.9) takes the form of (2.4) with $\sigma^{\ell}_{22}(X')$ given by (3.6). The system of three equations (2.3), (2.4), and (3.7) is a system of three linear algebraic equations for determination of $\sigma_{22}(c)$, $\sigma'_{22}(c)$ and k_1^{eff}. Substitutions of (2.3) and (3.7) into (3.6), and (3.6) into (2.4) result in the linear equation for K_1^{eff} with the solution (2.14). In this problem $q = q_o + q_1$,

where

$$q_o = \frac{f_o(\ell,\delta)}{\pi\sqrt{2(\ell'+\delta')}}, \text{ and}$$

$$q_1 = \frac{f_1(\ell,\delta)}{\pi\sqrt{2(\ell'+\delta')}}\cdot\frac{1}{4(\ell'+\delta')} \quad (3.8)$$

(both q_o and q_1 are nondimensional) with

$$f(\ell,\delta) = \int_{-1}^{1} \sqrt{\frac{1+X}{1-X}}\; I_o(C',X')dX \tag{3.9}$$

$$f_1(\ell,\delta) = \int_{-1}^{1} \sqrt{\frac{1+X}{1-X}}\; [I_1(C',X') + I_o(C',X')(C'-X')]dX$$

where

$$I_o(C'X') = \ell \int_{C-\ell}^{C+\ell} \frac{e(\xi)}{(\xi-X)^2}\, d\xi = \pi\left(\frac{1}{\sqrt{1-(\frac{\ell}{X-C})^2}^{\,2}} - 1\right) \tag{3.10}$$

and

$$I_1(C',X') = \ell \int_{C-\ell}^{C+\ell} \frac{e(\xi)}{\xi - X} d\xi = \pi(X' - C')\left(\sqrt{1-(\frac{\ell}{X-C})^2} - 1\right) \tag{3.11}$$

The coefficients q_o and q_1, are responsible for increase of effective stress intensity factor k_1^{eff} due to the elliptic microcrack opening and due to the linear deviation of elliptic shape, correspondingly.

General superposition formula (1.2) gives the resulting stress field in the form

$$\underset{\sim}{\sigma}(X) = K_1^{eff}\left\{\frac{\underset{\sim}{\phi}[\theta(X)]}{\sqrt{2\pi r(\underset{\sim}{X})}} + \right.$$

$$\left.\frac{4\ell}{E}\cdot\frac{1}{\sqrt{2\pi(\ell+\delta)}}\underset{\sim}{T}_X\int_{C-\ell}^{C+\ell}\left[1-\frac{\xi-C}{4(\ell+\delta)}\right] e(\xi)\underset{\sim}{n}(C).\underset{\sim}{\Phi}(\xi,\underset{\sim}{X})d\xi\right\} \tag{3.12}$$

Formula (3.12) has the same structure as (2.15). The difference

appears because of the second term in brackets in the integrand. This term represents the correction to constant approximation due to the first term of the expansion of the resulting stress field on the microcrack (3.1). The graph of $\frac{K_1^{eff}}{K_1^0}$ for linear approximation is given on Figure 13 together with the k_1^0 for constant approximation. Linear approximation is represented by the upper curve.

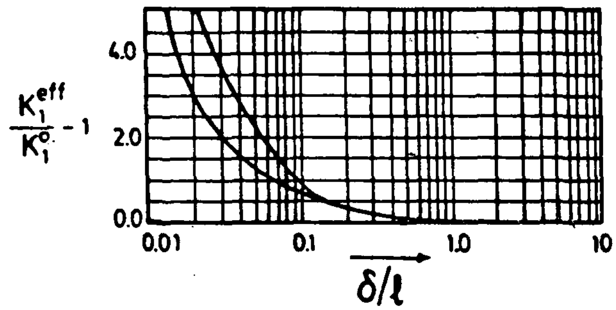

FIGURE 13

The deviation of the upper curve becomes significant for small δ/ℓ, i.e., when the distance between the macro and microcrack becomes small.

It should be noted that the solution obtained above gives low estimate for both k_1^{eff} and the stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ in (3.12). This is illustrated by Figure 14, where our solution corresponds to the approximation of the resulting stress field component $\sigma_{22}(x)$ by the tangent at the center of the microcrack.

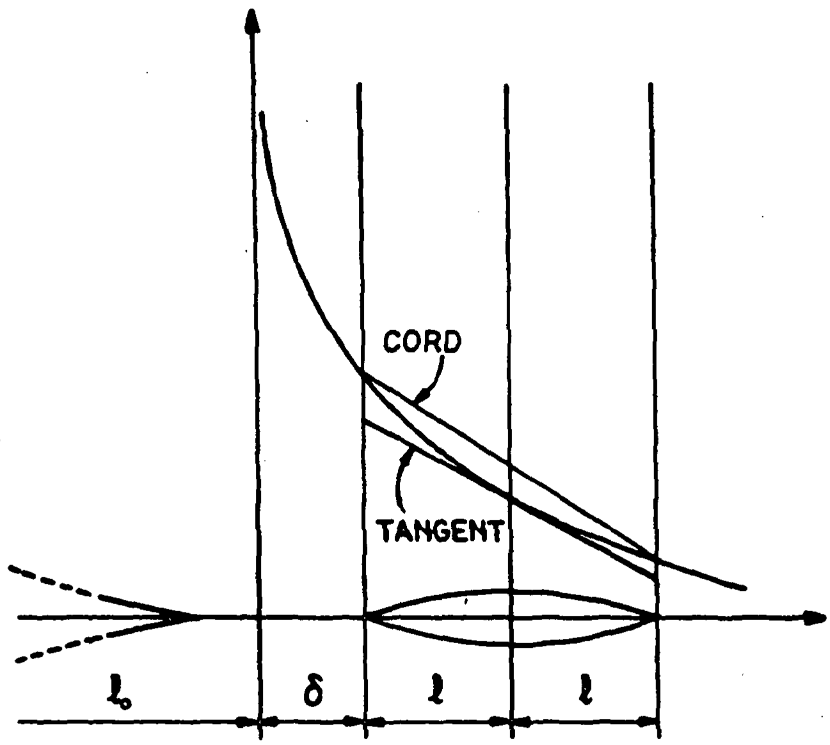

FIGURE 14

The results of calculations by the exact formula from [38] together with the upper (i.e., the approximation of the resulting stress field by the cord drawn through the end points of the micro-crack) and the lower bounds are shown in Figure 15. The formula from [38] employed for the evaluation of the ratios K_1^{eff}/K_1^o in the coordinate system of Figure 15 may be written as follows

$$\frac{K^{eff}}{K_1^o} = -\frac{1}{\sqrt{\delta(\delta+2\ell)}}\ (\ell_o+\delta+3\ell)\ (2\ell_o+\delta+2\ell)\ \frac{\Pi(n,K)}{F(K)} -$$

$$-(2\ell_o+\delta+2\ell)^2\ \frac{I(n,K)}{F(K)} - (2\ell+\delta)(\ell_o+\ell)\}$$

where
$$F(K) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - K^2 \sin^2 t}}$$

and
$$\Pi(n,K) = \int_0^{\frac{\pi}{2}} \frac{dt}{(1 + n\sin^2 t)\sqrt{1 - K^2 \sin^2 t}}$$

are the complete elliptic integrals of the first and third kind, respectively, and

$$I(n,K) = \int_0^{\frac{\pi}{2}} \frac{dt}{(1 + n\sin^2 t)^2 \sqrt{1 - K^2 \sin^2 t}}$$

with

$$n = \frac{2\ell_o}{X + 2\ell}, \quad \text{and} \quad K^2 = \frac{2\ell_o \ell}{(2\ell_o + \delta)(\delta + 2\ell)}$$

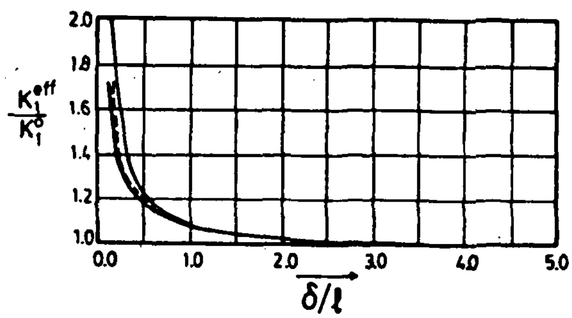

FIGURE 15

The lower curve in Figure 15 represents the tangent approximation, the dashed curve corresponds to the exact solution, and the upper curve represents the cord approximation.

Thus, for the microcrack (small in comparison to the macrocrack) located within the range of the macrocrack asymptotic stress field

(2.1) the solution by means of linear approximation obtained above is in a good agreement with the exact solution.

2. Quadratic and Cubic Approximations

This section is a direct continuation of the previous one; the quadratic and cubic approximations of the resulting stress field $\sigma(x)$ will be considered. Our goal is to develop the method of constructing the higher order approximations and possibly to represent the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ as an infinite series. The latter will be done in the next section. It will be shown also that under certain assumptions about the character of the stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ this series represents the exact solution to the problem.

The elastic stress field on the microcrack is assumed to be of the form

$$\underset{\sim}{\sigma}(X') = \underset{\sim}{\sigma}(C') + \underset{\sim}{\sigma}'(C')(X'-C') + \sigma''(C')\frac{(X'-C')^2}{2!} + \sigma'''(C')\frac{(X'-C')^3}{3!} \tag{3.13}$$

$$X \in (C-\ell,\ C+\ell)$$

where $\underset{\sim}{\sigma}'(c')$, $\underset{\sim}{\sigma}''(c')$ and $\underset{\sim}{\sigma}'''(c')$ are the derivatives of the stress field $\underset{\sim}{\sigma}(x')$ at point c'. By Willis's theorem [35] the crack opening displacement may be represented as a third degree polynomial superimposed on elliptic crack opening, so that

$$\underset{\sim}{b}(X') = [b_o + b_1(X'-C') + b_2\frac{(x'-C')^2}{2!} + b_3\frac{(X'-C')^3}{3!}]\ 4\ (X')\underset{\sim}{n}(C') \tag{3.14}$$

The boundary conditions (1.7) have the form (2.3) (i.e. $\sigma_{22}(c) = K_1^{eff}\frac{1}{\sqrt{2\pi(\ell+\delta)}}$), which leads to the conditions

$$\sigma'_{22}(C) = K_1^{eff} S'(\ell+\delta),$$

$$\sigma''_{22}(C) = K_1^{eff} S''(\ell+\delta),$$

$$\sigma'''_{22}(C) = K_1^{eff} S'''(\ell+\delta) \tag{3.15}$$

where

$$S(\underset{\sim}{X}) = \frac{1}{\sqrt{2\pi r(\underset{\sim}{X})}}$$

(all the derivatives s', s'', s''' are of the same units as S(x)). The stress field (3.13) must be expressed in terms of crack opening coefficients b_k by means of (1.6). This gives for $\sigma_{22}(X')$

$$\sigma_{22}(X') = \sum_{n=0}^{3} \sigma_{22}^{(n)}(C') \frac{(X'-C')^n}{n!} =$$

$$\underset{\sim}{n}(X')\underset{\sim}{n}(X')\underset{\sim}{T}_X \int_{C-\ell}^{C+\ell} \sum_{K=0}^{3} b_K \frac{(\xi'-C')^K}{K!} e(\xi')\underset{\sim}{n}(C).\underset{\sim}{\Phi}(\xi',X')d\xi \tag{3.16}$$

$$X \in (C-\ell,\ C+\ell)$$

The righthand part of (3.16) has to be a third degree polynomial, and by comparing the coefficients on the right and on the left of (3.16) the relationship between $\sigma_{22}^{(n)}(c)$ and b_n may be established. This is linear relationship and, consequently, can be written as follows in variant form

$$\{\sigma\} = \{B\}^{-1} \{b\} \tag{3.17}$$

or in component form as

$$\sigma_{22}^{(n)}(C) = \sum_{K=0}^{3} B_{nK}^{-1} b_K$$

where $\{B\}^{-1}$ is transformation matrix and $\{\sigma\}$ and $\{b\}$ stand for vector-columns $\sigma_{22}^{(n)}(C)$: $\{\sigma_{22}, \sigma_{22}', \sigma_{22}'', \sigma_{22}'''\}$,

$$b_K: \{b_1, b_2, b_3, b_4\} \tag{3.18}$$

or, evaluating the integrals in (3.16):

$$\{B\}^{-1} = \frac{E}{\ell}\begin{pmatrix} 1 & 0 & \frac{1}{4}\ell'^2 & 0 \\ 0 & 2 & 0 & \frac{1}{3}\ell'^2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{3.19}$$

Note that the only two singular integrals present in (3.18) have been evaluated already in section 1, see formula (3.4). In the case of quadratic approximation the matrix degenerates into 3 x 3 matrix which can be obtained from by crossing out the fourth column and the fourth row. It can be easily checked that for a linear approximation the matrix degenerates into 2 x 2 which can be obtained by crossing out the third and the fourth columns and rows (see formula (3.5)).

In order to obtain the effective stress intensity factor k_I^{eff} and the stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ we need to solve the system of equations (3.17). The inverse matrix has the form

$$\{B\} = \frac{\ell}{E}\begin{pmatrix} 1 & 0 & -\ell'^2/3.4 & 0 \\ 0 & 1/2 & 0 & -\ell'^2/3.4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{3.20}$$

The column {b} of crack opening coefficients may be represented now as follows

$$\{b\} = \{B\}\ \{\sigma\} \tag{3.21}$$

Substitution of (3.15) into (3.21) gives {b} as a linear function of k_1^{eff}

$$\{b\} = K_1^{eff}\ \{B\}\ \{S\} \tag{3.22}$$

where a vector column {S} is represented by the components $S(\ell + \delta)$, $S'(\ell + \delta)$, $S''(\ell + \delta)$, $S'''(\ell + \delta)$. Substitution of (3.14) with {b} given by (3.22) into (1.6) and the latter into equation for effective stress intensity factor (1.9) results in the linear equation for k_1^{eff}, with th solution (2.14). In order to carry out this procedure we substitute (3.14) into (1.6) at first, and after rearrangement of some terms obtain

$$\begin{aligned}\sigma_{22}^{\ell}(X') = -\frac{E}{\pi\ell}\Big\{ & \int_{-l}^{l} \frac{e(t)}{t'^2}\left[b_o + b_1(X'-C') + \frac{1}{2!}\,b_2(X'-C')^2 + \right. \\ & \left. + \frac{1}{3!}\,b_3(X'-C')^3\right]dt' - \int_{-l}^{l} \frac{e(t)}{t'}\left[b_1 + b_2(X'-C') + \right. \\ & \left. + \frac{1}{2}\,b_3(X'-C')\right]dt + \int_{-l}^{l} e(t)\left[\frac{1}{2}b_2 + b_3(X'-C')\right]dt' \\ & - \frac{1}{3}! \int_{-l}^{l} t'\,e(t)b_3 dt'\Big\}\end{aligned} \tag{3.23}$$

where $t = \xi - x$, $e(t) = \sqrt{1 - (\frac{t-x-c}{\ell})}$, and integrals are taken from $(c'-\ell'-x')$ to $(c + \ell' - x')$ introducing the notation

$$I_o(\ell',X') = \pi[\frac{1}{\sqrt{1 - (\frac{\ell}{X'-C'})^2}} - 1]$$

$$I_1(\ell',X') = \pi(X'-C')[\ 1 - (\frac{\ell'}{X'-C'})^2 - 1]$$

$$I_2(\ell',X') = -\int_{C'-\ell'-X'}^{C'+\ell'-X'} e(t)dt' = -\frac{\pi\ell'^2}{2}$$

$$I_3(\ell',X') = -\int_{C'-\ell'-X'}^{C'+\ell'-X'} t'\,e(t)dt' = \frac{\pi\ell'^2}{2}(X'-C')$$

(note that I_o and I_1 are the same integrals as in section 1, formulas (3.10) and (3.11)) formula (3.23) may be rewritten as follows:

$$\sigma_{22}^{\ell}(X) = -\frac{E}{\pi\ell}\{I_o b_o + [(X'-C')I_o + I_1]b_1 + \frac{1}{2!}[(X'-C')^2 + 2(X'-C')I_1 + I_2]b_2 + \frac{1}{3!}[(X'-C')^3 I_o + 3(X'-C')^2 I_1 + 3(X'-C')I_2 + I_3]\} \tag{3.23a}$$

Substitution of (3.22) into (3.23a) results in the following expression

$$\sigma_{22}^{\ell}(X) = -K_1^{eff}\ \frac{1}{\Pi}\{I_o(S - \frac{\ell'^2}{3\cdot4}S'') + \frac{1}{2}(S' - \frac{\ell'^2}{2\cdot3\cdot4}S''')\cdot$$

$$\cdot[(X'-C')I_o + I_1] + \frac{1}{2!3}S''[(X'-C')^2 I_o + 2(X'-C')I_1 + I_2] + \tag{3.23b}$$

$$+ \frac{1}{3!4} S'''[(X' - C')^3 I_o + 3(X' - C')^2 I_1 + 3(X' - C')I_2 + I_3]\} \qquad (3.23b)$$

Substituting (3.23b) into (1.9), results in

$$K_1^{eff} = k_1^o + k_1^{eff} (q_o + q_1 + q_2 + q_3)$$

where $k_1^{eff} q_o$, $k_1^{eff} q_1$, $k_1^{eff} q_2$, and $k_1^{eff} q_3$ represent the increments of effective stress intensity factor k_1^{eff}, resulting from constant, linear, quadratic and cubic approximations, respectively. The solution of the last equation appears in the form (2.14) with

$$q = \sum_{i=0}^{3} q_i \qquad (3.24)$$

Each of q_i - is defined by the structure of (1.9) and (3.23b) and may be written in the form

$$q_o = \frac{1}{\pi} \cdot \frac{\ell_o}{\pi} \left(S - \frac{\ell'^2}{3 \cdot 4} S''\right) \int_{-1}^{1} \sqrt{\frac{1 + X'}{1 - X'}}\, I_o(\ell', X')dX'$$

$$q_1 = \frac{1}{\pi} \cdot \frac{\ell_o}{\pi} \left(\frac{1}{2}S' - \frac{\ell'^2}{2 \cdot 3 \cdot 4} S'''\right) \int_{-1}^{1} \sqrt{\frac{1 + X'}{1 - X'}}\, [I_o(\ell', X') (X' - C') + I_1(\ell', X')]dX'$$

$$q_2 = \frac{1}{\pi} \cdot \frac{\ell_o}{\pi} \cdot \frac{S''}{2!3} \int_{-1}^{1} \sqrt{\frac{1 + X'}{1 - X'}}\, [(X' - C')^2 I_o(\ell', X') + 2(X' - C')I_1 (\ell', X') + I_2]dX'$$

$$q_3 = \frac{1}{\pi} \cdot \frac{\ell_o}{\pi} \cdot \frac{S'''}{3!4} \int_{-1}^{1} \sqrt{\frac{1 + X'}{1 - X'}}\, [(X' - C')^3 I_o + 3(X' - C')^2 I_1 + 3(X' - C')I_2 + I_3]dX'$$

The formulas above can be easily reduced to the ones in the linear approximation by setting $S''=S'''=0$.

Using general superposition forumla (1.2), formula for effective stress intensity factor (2.14) with q_i-th ($i = 1,2,3,4$) given by expressions above, and formula (3.23b) for the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ for cubic approximation may be written out. We do not do it here, but in the next section the general formula for the stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ is presented in a form of infinite series by the derivatives of dominating field at the microcrack center, i.e. at $r = \ell + \delta$.

In the considered case of cubic approximation of the elastic stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ on the microcrack all the calculations have been done exactly. In case of higher order approximations the complexity increases and only approximate calculations have been done. It should be noted that integrals of the type of I_0, I_1, I_2, etc. always can be evaluated in terms of elementary functions [37] and the procedure can be carried out in the case of higher approximations also.

The concluding remarks of Section 1, Chapter II are applicable to the problem of this section also. It means that the equation for the effective stress intensity factor may be solved by iterative procedure which corresponds to the expansion of (2.14) into geometric series. Physical meaning of the terms in geometric series remains the same.

Thus, the solution to the problem under consideration can be refined in two directions: higher order interactions taken into account, and higher degree of approximation of the stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$

involved.

3. The Series Solution (Two colinear Cracks)

In this section the procedure of obtaining higher order approximations developed in the previous section will be extended up to the P-th approximation, where P is an arbitrary number. It will be shown that for an analytic stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ the P-th approximation, becomes the exact solution in the limit $P \to \infty$. In the logic of development of this section we follow the pattern of the previous one. Thus, the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ on the microcrack ℓ will be represented by an expression

$$\sigma_{22}(X) = \sum_{K=0}^{P} \sigma_{22}^{(K)}(C) \frac{(X - C)^K}{K!}, \quad X \in (C-\ell, C+\ell) \tag{3.25}$$

then by Willis's theorem [35] the COD on the microcrack may be represented as follows

$$\underset{\sim}{b}(X) = 4\underset{\sim}{n}(C) \sum_{K=0}^{P} b_K \frac{(X - C)^K}{K!} (X) \tag{3.26}$$

Boundary conditions (1.7) and the ones obtained from them by means of differentiation (i.e., analogous to formula (3.15)) take the form:

$$\begin{aligned} \sigma_{22}(C) &= K_1^{eff} S(\ell + \delta) \\ \sigma'_{22}(C) &= K_1^{eff} S'(\ell + \delta) \\ \sigma_{22}^{(P)}(C) &= K_1^{eff} S^{(P)}(\ell + \delta) \end{aligned} \tag{3.27}$$

The relationship of the vector $\sigma_{22}^{(K)}(\ell) = \{\sigma\}$ to the vector $b_k = \{b\}$ is linear and given by (3.17), where the transformation matrix $\{B\}^{-1}$ is defined by the expression

$$\{B\}^{-1}\{b\} = \underset{\sim}{n}(X)\underset{\sim}{n}(X)\underset{\sim}{n}(C)\cdot\underset{\sim}{T}_x \int_{C-\ell}^{C+\ell} \sum_{K=0}^{P} \ell b_K \frac{(\xi - C)^K}{K!} e(\xi)\, \underset{\sim}{\Phi}(\xi, \underset{\sim}{X})d\xi \quad (3.28)$$

which is analogous to (3.18) of the previous section. Formula (3.28) (just as (3.18)) do not give the explicit form of transformation matrix. In the previous section, all the integrals in (3.18) have been evaluated and the resulting structure of $\{B\}^{-1}$ appeared as (3.19). In the general case under consideration higher order integrals in (3.28) were estimated with the help of the mean value theorem. The structure of transformation matrix $\{B\}^{-1}$ (and $\{B\}$) remains similar to the one given by (3.19) and (3.20), respectively).

In order to complete the scheme the equation for effective stress intensity factor (1.9) must be used; it is linear equation in view of (3.22), which preserves its form with $\{B\}^{-1}$ defined by (3.28) and the vector $\{s\}$ extended correspondingly. The solution of (1.9) has the usual form (2.14) with

$$q = \sum_{K=0}^{P} q_K \quad (3.29)$$

where

$$q_K = \frac{1}{\pi\sqrt{\pi\ell_o}} \sum_{i=0}^{P} B_{Ki}\, S_i \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \left\{ \int_{C-\ell}^{C+\ell} \frac{(\xi - C)^K}{K!} \cdot \frac{e(\xi)}{(\xi - X)^2} d\xi \right\} dX \quad (3.30)$$

$$X\ (C-\ell,\ C+\ell)$$

With effective stress intensity factor k_1^{eff} determined, the last step left is to use general superposition formula (1.2) for determining the stress field $\sigma(x)$.

$$\sigma(X) = K_1^{eff} \{\frac{\Phi[\theta(X)]}{\sqrt{2\pi r(X)}} + $$

$$+ \frac{1}{\pi} \sum_{n=0}^{P} \sum_{K=0}^{N} B_{nK} S_K T_X \int_{C-\ell}^{C+\ell} \frac{(\xi - C)^n}{n!} \underset{\sim}{e}(\xi) \underset{\sim}{n}(C) \cdot \underset{\sim}{\Phi}(\xi, \underset{\sim}{X}) d\xi\} \qquad (3.31)$$

Formulas (2.14) (with q defined by (3.29) and (3.30)), and (3.31) are the resulting ones for P-th approximation. The rest of this section is dedicated to the analysis of the structure of matrix {B} and the conditions under which formulas (2.14), (3.30) can be extended to exact results.

It should be noted that evaluation of the integrals in (3.28) is of vital importance because only that gives the explicit structure of transformation matrix. The integrals may be represented in the form

$$I_n(\ell, X) = - \ell \int_{C-\ell}^{C+\ell} (X - \xi)^{n-2} e(\xi) d\xi, \quad X \notin (C-\ell, C+\ell) \qquad (3.32)$$

(The first four of them are I_0, I_1, I_2, I_3 of the previous section.) Only I_0, I_1 are singular integrals and they have been evaluated already. All the integrals I_n are of the type $R(x, \; x^2 + \alpha x + \beta)$ where R is rational function and, therefore [37], can be integrated in terms of elementary functions. It is enough for our purposes, however, to estimate them by the mean value theorem

$$I_n(\ell, X) = \frac{\ell\, e(\xi_o)}{n - 1} \{[X - (C + \ell)]^{n-1} - [X - (C - \ell)]^{n-1}\}$$

where $e(x_0) \leq 1$, for $X_0 \in (C - \ell, c + \ell)$. The last formula can be rewritten as follows

$$I_n(\ell, X) = - \frac{\ell\, (X_0)}{n - 1} \sum_{m=0}^{n-1} C^m_{n-1} (X-C)^{n-m-1} \ell^m [(-1)^m - 1] \tag{3.33}$$

where C^m_{n-1} are the binominal coefficients.

Using the expression (3.33) with the exact results for the integrals I_0 - I_3 (section 2) the stress field on the microcrack can be expressed as follows:

$$\begin{aligned} \sigma_{22}(X) = - \frac{E}{\pi\ell} \{ & \sum_{n=0}^{N} (-\pi) \frac{bn}{n!} C^o_n (X-C)^n + \sum_{n=1}^{N} \pi C^1_n \frac{bn}{n!} (X-C)^n + \ldots \\ & +(-1)^{K+1} \sum_{n=K}^{N} C^K_n \frac{bn}{n!} \cdot \frac{e(X_0)}{K - 1} + \sum_{m=0}^{K-1} C^m_{K-1} \ell^m [(-1)^m - 1] (X-C)^{n-m-1} + \ldots \} \end{aligned} \tag{3.34}$$

By comparison of the coefficients at the same powers of (x - c) in the righthand part of (3.34) and its lefthand part in the form (3.25), we obtain the transformation matrix $\{B\}^{-1}$, from which matrix $\{B\}$ can be obtained.

The structure of both matrixes $\{B\}^{-1}$ and $\{B\}$ appear to be the same as in (3.19), (3.20), i.e.,

$$\{B\}^{-1} = \frac{E}{\ell} \begin{pmatrix} 1 & 0 & a_{02}\ell'^2 & 0 & a_{04}\ell'^4 & 0 & a_{06}\ell'^6 & \ldots \\ 0 & 2 & 0 & a_{13}\ell'^2 & 0 & a_{15}\ell'^4 & 0 & \ldots \\ 0 & 0 & 3 & 0 & a_{24}\ell'^2 & 0 & a_{26}\ell'^4 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \ldots \end{pmatrix} \tag{3.35}$$

and

$$\{B\} = \frac{E}{\ell}\begin{pmatrix} 1 & 0 & \beta_{02}\ell'^2 & 0 & \beta_{04}\ell'^4 & 0 & \beta_{06}\ell'^6 & \cdots \\ 0 & \frac{1}{2} & 0 & \beta_{13}\ell'^2 & 0 & \beta_{15}\ell'^4 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \beta_{24}\ell'^2 & 0 & \beta_{26}\ell'^4 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \tag{3.36}$$

where coefficients α_{02}, α_{04}, α_{06}, α_{13}, α_{15}, etc. and β_{02}, β_{04}, β_{06}, β_{13}, β_{15}, etc. can be calculated from (3.34). It should be noted that coefficients ij decrease by both indexes i and j.

The structure of transformation matrixes $\{B\}^{-1}$ and $\{B\}$ described above is not a result of approximate calcultions of integrals I_n. This can be shown by considering the integrals I_n: Notice that the product of binomial expansion of the term $(x - \xi)^{n-2}$ multiplied by elliptic COD $e(\xi)$ represents the sequence of odd and even terms. Integration within symmetric limits leaves only even terms.

Formulas (3.35) and (3.36) complete the consideration of the matrixes $\{B\}^{-1}$ and $\{B\}$. If the $\sigma_{22}(x)$ component of a stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ is analytic in the interval $(c - \ell, c + \ell)$, then (3.25) may be considered as its Taylor's expansion when $P \to \infty$. Correspondingly (3.26) is Taylor's expansion for $\underset{\sim}{b}(\underset{\sim}{x})$ and formulas (2.14) (with q_k given by (3.30)) and (3.31) represent the exact solution of the problem if the series in (3.30) and (3.31) converge.

The convergence in both formulas is defined by the behavior of the following series

$$\sum_{n \geq k} \sum_{k \geq 0} B_{nk} S_k \int_{c-\ell}^{c+\ell} \frac{(\xi - C)^n}{n!} \underset{\sim}{f}(\xi, \underset{\sim}{x}) d\xi \tag{3.37}$$

where $f(\xi, x) = \dfrac{e(\xi)}{(\xi - x)^2}$ is a scalar function for formula (3.29) and $f(\xi, x) = e(\xi)\, \underset{\sim}{n}(C) . \underset{\sim}{\Phi}(\xi, \underset{\sim}{x})$, where $x \epsilon (c - \ell, c + \ell)$ is a vector function for formula (3.31). Taking into account that

$$S_K = \left(\frac{1}{\sqrt{2\pi r}} \right)^{(K)} \Bigg|_{r = \ell + \delta} = \frac{(-1)^K (2K-1)!!}{\sqrt{2\pi(\ell+\delta)}\,(\ell+\delta)^K 2^K} , \tag{3.38}$$

for K>0

and that

$$\left| \int_{c-\ell}^{c+\ell} \frac{(\xi - c)^n}{n!} \underset{\sim}{f}(\xi, \underset{\sim}{x}) d\xi \right| \leq \frac{\ell^n}{n!} \left| \int_{c-\ell}^{c+\ell} \underset{\sim}{f}(\xi, \underset{\sim}{x}) d\xi \right| \tag{3.39}$$

(3.37) can be bounded as follows:

$$\left| \sum_{n \geq} \sum_{k \geq 0} B_{nk} S_k \int_{c-\ell}^{c+\ell} \frac{(\xi - c)^n}{n!} \underset{\sim}{f}(\xi, \underset{\sim}{x}) \right| \leq$$

$$\left| \frac{1}{\sqrt{2\pi(\ell+\delta)}} \sum_{n>k>0} \sum B_{nk} \frac{(2k-1)!!}{2^k n!} \cdot \frac{1}{(1 + \delta/\ell)^K} \int_{c-\ell}^{c+\ell} \underset{\sim}{f}(\xi, \underset{\sim}{x}) d\xi \right| \tag{3.40}$$

The series on the right of (3.40) is absolutely convergent and the rate of convergence depends on δ/ℓ : the bigger the ratio δ/ℓ, the better the convergence is. This result is understandable from a physical point of view; the small ratio indicates that either the microcrack is close to the main crack tip (i.e., $\delta \to 0$), or the microcrack length ℓ is large; in both cases many terms in the stress field expansion are needed. For a large ratio either the

microcrack is far from the main crack tip (i.e., δ is large), or the microcrack is small (i.e., $\ell \to 0$); in both cases the correction to the main crack dominating field is small and our series (3.37) converges rapidly.

As conclusion to this section we will formulate the above as the following statement: if a system of two cracks in linear elastic medium under the assumptions of plane problem and mode I loading located as on Figure 5, and the microcracks are imbedded in the dominant field of the main crack, then the analytic stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ can be represented as an infinite series (3.31) where: 1. the effective stress intensity factor k_1^{eff} is given by (2.14), (3.29) and (3.30), 2. the first term represents the dominating asymptotic stress field of the main crack, 3. the second term represents the series by the derivatives of the dominant field S_k taken at the center of the microcrack with coefficients which depend on small crack length and location, 4. the series is absolutely convergent and the rate of convergence depends on the ratio of the distance between two adjacent crack tips δ to the smaller crack length ℓ.

The obtained result is an exact series solution to the problem of this chapter. It should be noted, that the exact solution to this problem has been obtained in [38] by means of complex variable techniques. It will be shown in the next chapter that the method suggested in this section may be generalized and the exact solution to the problem of a macrocrack surrounded by an array of arbitrarily located microcracks of arbitrary lengths and orientations can be obtained.

CHAPTER IV

The Multiple Crack Interaction Problem

1. Stress Field in a Neighborhood of a Macrocrack Tip Surrounded by an Array of Microcracks

In this section we consider a plane elastostatic problem of the interaction of a macrocrack under mode I tensile loading with an array of rectilinear microcracks at the macrocrack tip.

Let us introduce the following notation,

1. $$\underset{\sim}{\sigma}_o(\underset{\sim}{X}) = K_1^o \frac{\underset{\sim}{\phi}[\theta(\underset{\sim}{X})]}{\sqrt{2\pi r(\underset{\sim}{X})}} \tag{4.1}$$

is the asymptotic stress field of a macrocrack of length $2\ell_o$ with K_1^o - as stress intensity factor $\underset{\sim}{\phi}[\theta(\underset{\sim}{x})]$ - as asymptotic angle distibution tensor, and $r(\underset{\sim}{x})$ - as a distance from macrocrack tip to the point $\underset{\sim}{x}$.

2. $$t_m(\underset{\sim}{X}^s) = n(X_o^s)\sigma^{(m)}(X_o^s) \tag{4.2}$$

is m-th derivative of traction vector $\underset{\sim}{t}$ in the direction of s-th microcrack evaluated at the center of the microcrack. The directional derivative of higher order must be undestood in the following sense: if $X_2^s - X_{20}^s = \alpha(x_1^s - x_{10}^s)$ is the equation of the rectilinear segment of s-th microcrack (see Figure 16), then the stress $\underset{\sim}{\sigma}(x_1^s, x_2^s)$ on that segment can be represented as $\underset{\sim}{\sigma}[X_1^s, X_{20}^s + \alpha(X_1^s - X_{10}^s)]$ - i.e. a function of x_1^s coordinate only. Then the directional derivative takes the form

$$\underset{\sim}{\sigma}^{(m)}(\underset{\sim}{X}_o^s) = \frac{d}{dX_1^m} \underset{\sim}{\sigma}[X_1^s, X_{20}^s + \alpha(X_1^s - X_{10}^s)] \tag{4.3}$$

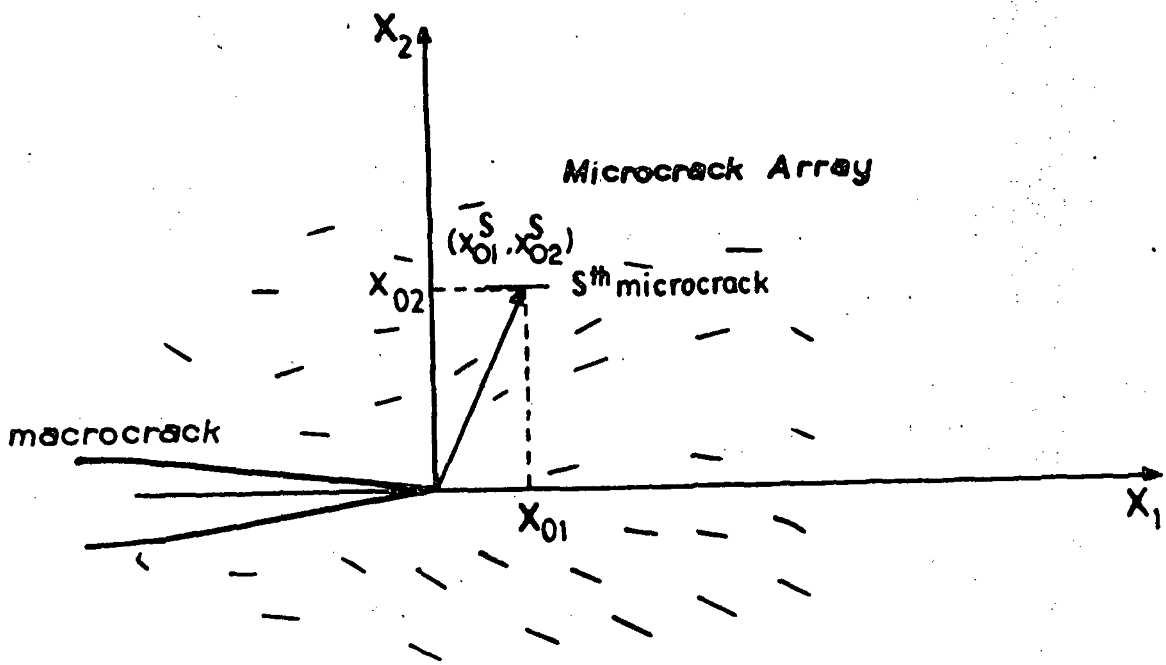

FIGURE 16

3. $b_k^{(n)} = \{b\}$

stands for the set of vector coefficients of a polynomial which is superimposed on the elliptic crack opening of $\underset{\sim}{k}$-th microcrack $e(\xi^k)$ (see formula (4.14)).

4. $\{\underset{\sim}{B}\} = B_{mn}(X_o^s, X_o^K)$ (4.4)

is a linear operator which depends on the positions of centers of two microcracks x_o^s, x_o^k, and characterizes the influence m-th derivative of s-th microcrack stress field on n-th derivative of k-th microcrack opening displacement coefficient $\{\underset{\sim}{b}\}$. The linear operator $\{\underset{\sim}{B}\}$ is a second rank tensor (subscripts in (4.2) and in all the operators below are not tensorial). The definition of the linear operator $\{\underset{\sim}{B}\}$ is given below.

5. $\{\underset{\sim}{A}(\underset{\sim}{X})\} = \underset{\sim n}{A}(\underset{\sim}{X}^K) = \underset{\sim}{T}_X \int_{\ell_K} \frac{(\xi^K - \xi_o^K)^n}{n!} e(\xi^K)\ \underset{\sim}{\phi}(\underset{\sim}{\xi}^K, \underset{\sim}{X})d\xi^K$ (4.5)

is a matrix which characterizes the stress associated with n-th derivative of the microcrack opening coefficient $\{\underset{\sim}{b}\}$ of k-th microcrack at arbitrary point $\underset{\sim}{x}$. Matrix $\{\underset{\sim}{A}(\underset{\sim}{x})\}$ is the matrix of the microcrack array. In formula (4.5) T_x - is stress operator transforming the displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ into stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$, $e(\xi)$ - is a unit elliptic crack opening displacement, $\underset{\sim}{\phi}(\underset{\sim}{\xi}, \underset{\sim}{x})$ - is the second Green's tensor defined in Appendix I, ξ^k is a coordinate on k-th microcrack of length $2\ell_k$; matrix $\{\underset{\sim}{A}(\underset{\sim}{x})\}$ is a third rank tensor,

6. $\{\underset{\sim}{A}^o\} = \underset{\sim n}{A}^o(\underset{\sim}{X}^K)$ (4.6)

is a linear operator which characterizes the increase of stress intensity factor K_1^o due to stresses associated with microcrack opening coefficients $\{\underset{\sim}{b}\}$ of the array. The linear operator $\{\underset{\sim}{A}^o\}$ is a vector. The definition of $\{\underset{\sim}{A}^o\}$ is given below.

Using the notation above, the following statement can be proved: If,

1. The microcrack array has been located in a close vicinity of a macrocrack tip and the characteristic linear dimension of the array is small in comparison to the macrocrrack length (i.e., small scale microcracking model), and

2. The resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ induced by the macrocrack and the rest of microcracks on the line of each microcrack can be approximated by the polynomial function.

Then the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ can be fully characterized

by the asymptotic stress field of the macrocrack $\underset{\sim}{\sigma}_0(x)$, the values of its derivatives $\sigma_0^{(m)}(x_0)$ in the directions of microcracks evaluated at the centers of microcracks, and the second Green's tensor $\phi(\xi, x)$ as follows:

$$\underset{\sim}{\sigma}(\underset{\sim}{X}) = (\{\underset{\sim}{1}\} + \{\underset{\sim}{B}\}\ \{\underset{\sim}{t}\}\ \{\underset{\sim}{A}^o\})\underset{\sim}{\sigma}_o(\underset{\sim}{X}) + \{\underset{\sim}{B}\}\ \{\underset{\sim}{t}\}\ \{\underset{\sim}{A}(\underset{\sim}{X})\} \tag{4.7}$$

or in index notation

$$\underset{\sim}{\sigma}(\underset{\sim}{X}) = [\underset{\sim}{1} + \sum_{m,n,s,K} B_{mn}(\underset{\sim}{X}_o^s, \underset{\sim}{X}_o^K)\underset{\sim}{t}_m(\underset{\sim}{X}^s)\underset{\sim}{A}^o(\underset{\sim}{X}^K)]\underset{\sim}{\sigma}_o(\underset{\sim}{X}) + \sum_{m,n,s,K} \underset{\sim}{B}_{mn}(\underset{\sim}{X}_o^s, \underset{\sim}{X}_o^K)\underset{\sim}{t}_m(\underset{\sim}{X}_o^s)\ \underset{\sim}{A}_n(\underset{\sim}{X}^K) \tag{4.7a}$$

<u>Proof</u>

The elastic stress field of the system of cracks under consideration can be represented by means of general superposition formula

$$\underset{\sim}{\sigma}(\underset{\sim}{X}) = \underset{\sim}{\sigma}(\underset{\sim}{X}) + \sum_{i=i} \underset{\sim}{\sigma}_i(\underset{\sim}{X}) \tag{4.8}$$

where

$$\underset{\sim}{\sigma}(X) = K_1^{eff}\frac{\underset{\sim}{\phi}[\theta(\underset{\sim}{X})]}{\sqrt{2\pi r(\underset{\sim}{X})}} \tag{4.9}$$

and

$$\underset{\sim}{\sigma}_i(\underset{\sim}{X}) = \underset{\sim}{T}_X \int_{\ell_i} \underset{\sim}{b}_i(\xi)\cdot \underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{X})d\xi \tag{4.10}$$

with $b_i(\xi)$ as a double layer potential density which can be interpreted as a crack opening displacement. Formula (4.9) represents dominant stress field of the main crack with K_1^{eff} - as effective stress intensity factor which reflects the impact of the microcrack array on the macrocrack. It appears as an unknown quantity in (4.9) along with unknown components of vectors $\underset{\sim}{b}_i(\underset{\sim}{\xi})$, where i = 1, 2, ... N.

Equilibrium equations are automatically satisfied for the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ in the form (4.8). The stress field of microcracks $\sum_{i=1}^{N} \underset{\sim}{\sigma}_i(x)$, which is defined by (4.10), satisfies the equilibrium equations because of the properties of the second Green's tensor $\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{x})$. The dominant stress field $\underset{\sim}{\hat{\sigma}}(\underset{\sim}{x})$, given by (4.9) satisfies the equilibrium equations because of the properties of the asymptotic crack tip solution.

The equations to be satisfied are the boundary conditions on the macrocrack and each of the microcracks. For the small scale model, the boundary condition on a macrocrack can be substituted by the equation for effective stress intensity factor K_1^{eff} which fully determines the asymptotic stress field of the macrocrack.

$$K_i^{eff} = K_1^o + \frac{1}{\sqrt{\pi\ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}}\, \underset{\sim}{n}(\underset{\sim}{X})\underset{\sim}{n}(\underset{\sim}{X}) \sum_{i=1}^{N} \underset{\sim}{\sigma}_i(\underset{\sim}{X})dX \tag{4.11}$$

Boundary conditions on microcracks appear to form a system of 2N

singular integral equations for 2N unknown components of N vectors of double layer potential densities $\underset{\sim}{b}(\underset{\sim}{x})$ (K = 1, 2, ... N).

$$\underset{\sim}{n}_i\{\hat{\underset{\sim}{\sigma}}(\underset{\sim}{X}_i) + \sum_{\substack{K=1\\K\neq i}}^{N} \underset{\sim}{T}_X \int_{\ell_K} \underset{\sim}{b}_K(\xi)\cdot\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{X}_i)d\xi + \underset{\sim}{T}_X \int_{\ell_i} \underset{\sim}{b}_i(\underset{\sim}{\xi})\cdot\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{X}_i)d\xi\} = 0 \qquad (4.12)$$

where x_i - is a coordinate on i-th microcrack. Equations (4.11) , (4.12) constitute a system for determination of 2N + 1 unknowns K_1^{eff} and the components of N vectors $\underset{\sim}{b}_k(\underset{\sim}{\xi})$.

In order to solve the system of equations (4.11), (4.12) let us represent the polynomial stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ on k-th microcrack by:

$$\underset{\sim}{\sigma}(\underset{\sim}{X}^K) = \sum_{n=0}^{P} \underset{\sim}{\sigma}_o^{(n)} (X_o^K) \frac{(X^K - X_o^K)^n}{n!} \qquad (4.13)$$

Formula (4.13) and Willis's theorem [35] permit to represent the COD $b(x^k)$ as a following polynomial

$$\underset{\sim}{b}(\underset{\sim}{X}^K) = \sum_{n=0}^{P} \underset{\sim}{b}_K^{(n)} (\underset{\sim}{X}_o^K) \frac{(X^K - X_o^K)^n}{n!} e(X) \qquad (4.14)$$

In view of (4.13) and (4.14) the boundary conditions (4.12) may be written as follows:

$$\underset{\sim}{n}(\underset{\sim}{X}^K)\underset{\sim}{T}_X\int_{\ell_K} \sum_{n=0}^{P} \underset{\sim}{b}^{(n)} (\underset{\sim}{\xi}_o^K)\cdot \frac{(\xi^K - \xi_o^K)^n}{n!} e(\xi^K)\,\underset{\sim}{\Phi}(\underset{\sim}{\xi}_k, \underset{\sim}{X}_K)d\xi =$$

$$-\sum_{n=0}^{P} \sum_{s=1}^{N} \underset{\sim}{n}(\underset{\sim}{X}_o^K)\cdot\underset{\sim}{T}_X \int_{\ell_s} \underset{\sim}{b}^{(n)} (\underset{\sim}{\xi}_o^s) \frac{(\xi^s - \xi_o^s)^n}{n!} e(\xi^s)\,\underset{\sim}{\Phi}(\underset{\sim}{\xi}^s, \underset{\sim}{X}^K)d\xi - \qquad (4.15)$$

$$- K_1^{eff}\ \underset{\sim}{S}(\underset{\sim}{X}^K)$$

where $\underset{\sim}{S}(\underset{\sim}{X}^K) = \underset{\sim}{n}(\underset{\sim}{X}^K) \dfrac{\underset{\sim}{\phi}[\theta(\underset{\sim}{X})]}{\sqrt{2\pi r(\underset{\sim}{X})}}$

Expression on the left of (4.15) represents traction on K-th microcrack which, by assumption, may be represented as a polynomial. (The lefthand part of (4.15) contains singular integral, when n=0, which converges in Cauchy's sense). Consequently, the righthand part of (4.15) (i.e., the traction on K-th microcrack induced by the rest of the microcracks and the dominant field) is a polynomial also; then (4.15) can be rewritten in the form of a system of equations for the coefficients of the polynomials of the lefthand and righthand parts of (4.15). The procedure described above can be carried out by differentiation of (4.15) at point $\underset{\sim}{x}_o^k$ in the direction of k-th microcrack P times. The differentiation results in the following equation:

$$\sum_{n=0}^{P} \sum_{K=1}^{N} \underset{\sim}{T}_{mn}(\underset{\sim}{X}_o^s, \underset{\sim}{X}_o^K)\underset{\sim}{b}_K^n = - \sum_{n=0}^{P} \sum_{K=1}^{N} \underset{\sim}{H}_{mn}(X_o^s, X_o^K)\underset{\sim}{b}_K^n - K_1^{eff} \underset{\sim}{S}_m(\underset{\sim}{X}_o^s) \tag{4.16}$$

where $\underset{\sim}{S}_m(\underset{\sim}{x}_o^s)$ is m-th derivative on s-th microcrack, m = 0, 1, 2, ... P, S = 1, 2, ... N. $\underset{\sim}{b}_k^n = \underset{\sim}{b}^{(n)}(x_o^k)$. The linear operator $\underset{\sim}{H}_{mn}(\underset{\sim}{x}_o^s \underset{\sim}{x}_o^k)$ is given by the following expressions:

$$\underset{\sim}{H}_{mn}(\underset{\sim}{X}_o^s, \underset{\sim}{X}_o^K) = \begin{cases} 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . & \text{when } s = k \\ \underset{\sim}{T}_{\dot{X}} \displaystyle\int_{\ell_s} \frac{(\xi^s - \xi^s)^n}{n!} e(\underset{\sim}{\xi}^s)\, \underset{\sim}{\Phi}^{(m)}(\underset{\sim}{\xi}^s, \underset{\sim}{X}_o^K)\cdot \underset{\sim}{n}(\underset{\sim}{X}_o^K)d\xi^s & \text{when } s \neq k \end{cases} \tag{4.17}$$

taken at point $x = x_o^k$

where $\Phi^{(m)}(\underset{\sim}{\xi}^s, \underset{\sim}{x}_o^k)$ is m-th directional derivative of Green's tensor at point k taken in direction of k-th microcrack defined by (4.4). Similarly, the linear operator $I_{mn}(\underset{\sim}{x}_o^s, \underset{\sim}{x}_o^k)$ is given by the expres-

sion:

$$\underset{\sim}{I}_{mn}(\underset{\sim}{X}_o^s, \underset{\sim}{X}_o^K) = \begin{cases} 0 \ldots\ldots\ldots\ldots\ldots & \text{when } s \neq k \\ \underset{\sim}{T}_{\underset{\sim}{X}} \displaystyle\int_{\ell_s} \frac{(\xi^s - \xi^s)^n}{n!} e(\underset{\sim}{\xi}^s)\, \underset{\sim}{\Phi}^{(m)}(\underset{\sim}{\xi}^s, \underset{\sim}{X}_o^K)\cdot \underset{\sim}{n}(\underset{\sim}{X}_o^K) d\xi^s & \text{when } s = k \end{cases} \quad (4.18)$$

The set of elements $\underset{\sim}{b}_k^n$ in (4.16) constitutes a P x N matrix. In symbolic notation equation (4.16) may be written as follows:

$$\{\underset{\sim}{I}\} \cdot \{\underset{\sim}{b}\} = - \{\underset{\sim}{H}\}\cdot\{\underset{\sim}{b}\} - K_1^{eff} \cdot \{\underset{\sim}{s}\} \quad (4.16a)$$

together with equation (4.11) for stress intensity factor k_1^{eff} boundary conditions on microcracks (4.12) form a system of (P x N + 1) linear algebraic equations with P x N unknown components of matrix {b} and unknown quantity K_1^{eff}. Substitution of formula (4.10) with $\underset{\sim}{b}_i(\underset{\sim}{\xi})$ given by (4.14) into (4.11) results in

$$K_1^{eff} = K_1^o + \{A^o\} \cdot \{b\} \quad (4.19)$$

where linear operator $\{\underset{\sim}{A}^o\}$ is given by the expression

$$\{\underset{\sim}{A}^o\} = \frac{1}{\sqrt{\pi \ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + X}{\ell_o - X}}\, \underset{\sim}{n}(\underset{\sim}{X}) \sum_{n=1}^{P} \sum_{s=o}^{N} \underset{\sim}{H}_{on}(\underset{\sim}{X}^s, \underset{\sim}{X}) dx \quad (4.20)$$

with $H_{on}(x_o^s, x)$ defined by (4.17). Formula (4.20) defines linear operator $\{\underset{\sim}{A}^o\}$ of (4.5). Operator $\{\underset{\sim}{A}^o\}$ characterizes the impact of a microcrack array on the main crack.

Substitution of (4.19) into (4.16a) gives a system of linear

algebraic equations for the determination of $\{\underset{\sim}{b}\}$

$$\{\underset{\sim}{I}\}\ \{\underset{\sim}{b}\} = -\ (\{\underset{\sim}{H}\} + \{\underset{\sim}{S}\}\ \{\underset{\sim}{A}^{o}\})\ \{\underset{\sim}{b}\}\ -\ \{\underset{\sim}{t}\} \tag{4.21}$$

The last system of equations yields the obvious solution

$$\{\underset{\sim}{b}\} = \{\underset{\sim}{B}\}\ \{\underset{\sim}{t}\} \tag{4.22}$$

where

$$\{\underset{\sim}{B}\} = -\ (\{\underset{\sim}{I}\} + \{\underset{\sim}{H}\} + \{\underset{\sim}{S}\}\ \{\underset{\sim}{A}^{o}\})^{-1} \tag{4.23}$$

Formula (4.23) defines the linear operator $\{\underset{\sim}{B}\}$ of (4.4).

In index notation (4.20) takes the form

$$\underset{\sim K}{b}^{n} = \sum_{s=1}^{N} \sum_{m=o}^{P} \underset{\sim mn}{B}(\underset{\sim o}{X}^{s},\ \underset{\sim o}{X}^{K})\ \underset{\sim m}{t}(\underset{\sim o}{X}^{s}) \tag{4.22a}$$

Formula (4.22) represents $\{\underset{\sim}{b}\}$ as a linear function of directional derivatives of the asymptotic stress field $\underset{\sim}{\sigma}_{0}(\underset{\sim}{x})$ $(\underset{\sim}{n}.\ \underset{\sim}{\sigma}_{0}(\underset{\sim}{x}) = K_{1}^{o}$ $\underset{\sim}{s}(\underset{\sim}{x})$. Substitution of (4.22) into (4.14), (4.14) into (4.10), and (4.19) into (4.9), with subsequent substitution of (4.9) and (4.10) into (4.8) furnishes (4.7). This completes the proof.

Thus, taking into account the analytic character of the solution of a plane problem of elastostatics [33], the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ can be approximated by a polynomial as closely as desired. Consequently the solution (4.7) obtained above can be made as close as

desired to the exact one.

It should be noted that each term in formula (4.7), for the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ can be given a clear physical interpretation; the first term represents dominating stress field of the main crack $\hat{\underset{\sim}{\sigma}}(\underset{\sim}{x})$ and the second term represents the stress field of the microcrack array embedded into the stress field of the macrocrack $\underset{\sim}{\sigma}_o(\underset{\sim}{x})$. Dominating stress field $\hat{\underset{\sim}{\sigma}}(\underset{\sim}{x})$, in turn, consists of two terms, first of them being the asymptotic stress field of the main crack $\underset{\sim}{\sigma}_o(\underset{\sim}{x})$, and the second term results from the impact of the microcrack array (embedded into asymptotic stress field) on the macrocrack.

The other remark concerns the method of solution of a system of equations (4.21). Actual construction of operator $\{\underset{\sim}{B}\}$ may present considerable difficulties for an extensive microcrack array. This is one of the reasons for suggesting an iterative procedure as an alternative to exact solution. Another reason is that an iterative procedure has clear physical meaning and, thus, gives an insight into the nature of solution (4.7), as will be shown below.

Iterative process for equation (4.21) can be arranged by multiplying it by $\{\underset{\sim}{I}\}^{-1}$ operator. This gives

$$\{\underset{\sim}{b}\}^{(n+1)} = - \{\underset{\sim}{I}\}^{-1} \left[\{\underset{\sim}{H}\} + \{\underset{\sim}{S}\}\{\underset{\sim}{A}^o\}\right] \{\underset{\sim}{b}\}^{(n)} - \{\underset{\sim}{I}\}^{-1} \{\underset{\sim}{t}\} \qquad (4.23)$$

where superscripts (n + 1) and (n) refer to the corresponding steps of iteration. Choosing $\{\underset{\sim}{b}\}^{(o)} = -\{I\}^{-1}\{t\}$ (i.e. microcrack opening coefficient $\{b\}^o$ which results from the main crack field only) as

zero approximation, we obtain

$$\{b\}^{(n)} = [\{1\} - \{D\} + \{D\}^2 - \{D\}^3 + \ldots + (-1)^n \{D\}^n]\{I\}^{-1} \{t\} \quad (4.24)$$

where $\{1\}$ is a unit operator, and

$$\{D\} = \{I\}^{-1} [\{H\} + \{S\}\{A^o\}] \quad (4.25)$$

The multiplication of linear operators must be understood in the following sense: e.g.,

$$\{I\}^{-1} \{H\} = \sum_{m,s} I^{-1}_{ms} (X^s_o, X^K_o) H_{mp} (X^s_o \; X^q_o) \quad (4.26)$$

Consequently, double iterated operator $\{D\}^2$ is represented by formula

$$D^2_{np} (X^K_o, X^q_o) = \sum_{m,s} D_{mn} (X^s_o, X^K_o) D_{mp} (X^s_o, X^q_o) \quad (4.27)$$

Substitution of (4.26) (with $\{b\}^o = - \{I\}^{-1} \{t\}$ into (4.7) after rearrangements can be written in the form

$$\sigma(X) = \sigma_o(X) - \{b\}^{(o)}\{A(X)\} + \{D\}\{b\}^{(o)}\{A(X)\} - \{D\}^2 \{b\}^{(o)}\{A(X)\} + \quad (4.28)$$

$$\ldots - \{b\}^o \{A\}^o \sigma_o(X) + \{D\}\{b\}^{(o)}\{A\}^o \sigma_o(X) - \{D\}^2 \{b\}^{(o)}\{A^o\}\sigma_o(X) + \ldots$$

Each consequent term in the last formula can be interpreted as follows: zero order term - $\sigma_o(x)$, i.e., asymptotic stress field of

a macrocrack; first order term - $\{\underset{\sim}{b}\}^{(o)}\{\underset{\sim}{A}(\underset{\sim}{x})\}$, i.e. stress field of the microcrack array (characterized by matrix $\{\underset{\sim}{A}(\underset{\sim}{x})\}$) embedded into asymptotic field $\underset{\sim}{\sigma}_0(\underset{\sim}{x})$ (note that $\{\underset{\sim}{b}\}^{(o)}$ represents the set of microcrack opening coefficients for the array embedded into asymptotic field $\underset{\sim}{\sigma}_0(\underset{\sim}{x})$); second order terms are characterized by triple products: $\{\underset{\sim}{D}\}\{\underset{\sim}{b}\}^{(o)}\{\underset{\sim}{A}(\underset{\sim}{x})\}$ and $\{\underset{\sim}{b}\}^{(o)}\{\underset{\sim}{A}^0\}\underset{\sim}{\sigma}_0(\underset{\sim}{x})$; the first term with the once-iterated operator $\{\underset{\sim}{D}\}$ gives the stress field generated by the microcrack array with microcracks subjected to stresses induced by other microcracks and the main crack, the microcrack and the main crack being embedded into asymptotic field; the second term with operator $\{\underset{\sim}{A}^0\}$ gives the stress field of the main crack subjected to the stresses induced by the microcrack array, the latter embedded into asymptotic stress field of the main crack.

Thus, double products in (4.28) account for I-st order interactions, triple product - for 2nd order interaction, etc. The coordinates above can be illustrated by the following diagram.

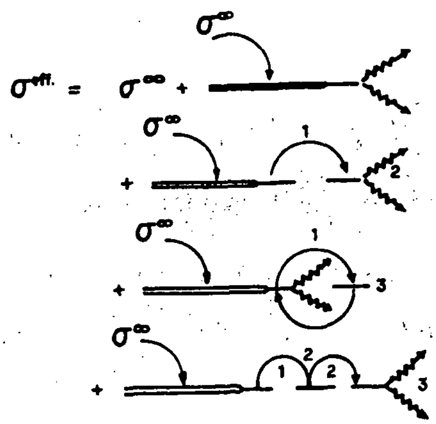

FIGURE 17

Thus, the stress field solution $\underset{\sim}{\sigma}(\underset{\sim}{x})$, given by (4.7), permits two different types of approximations:

1. Approximation by number of derivatives of the asymptotic stress field taken into account and 2. Approximation by number of physical interactions between microcracks taken into account.

2. Evalution of the J-Integral

In this section we evaluate the energy release rate J per unit crack layer extension. for the CL it can be expresed in the form of path-independent integral around the CL active zone V_A [25].

$$J_K = \int_{\Gamma} (f\delta_{Kj} - \sigma_{ij}u_{i,K})n_j d\Gamma \tag{4.29}$$

The active and inert zones V_A and V_I, the line of their separation $\Gamma(t)$ (i.e. the crack layer trailing edge [25]), and the contour Γ are presented in Figure 18.

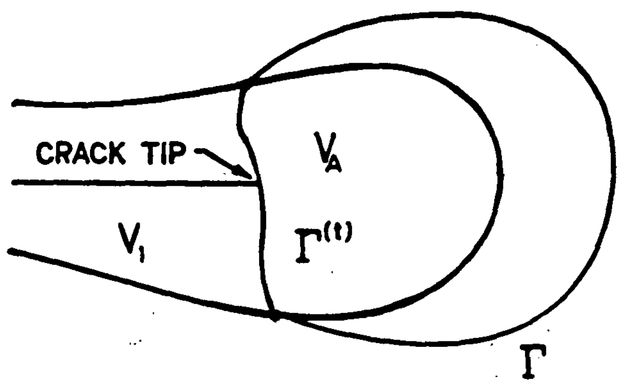

FIGURE 18

It is shown in [25] that J-integral given by (4.29) does not depend on the contour of integration Γ if the latter connects the end points of the trailing edge $\Gamma^{(t)}$ leaving the active zone V_A inside. The path-independent property of energy release rate J follows from the principle of minimum of strain energy with the additional condition of invariance of the strain energy with respect to translation of a CL. Consequently, the J-integral is path independent for homogeneous medium only. For a model of the CL described above (i.e., a macrocrack surrounded by an array of microcracks) Figure 18 must be substituted by Figure 19.

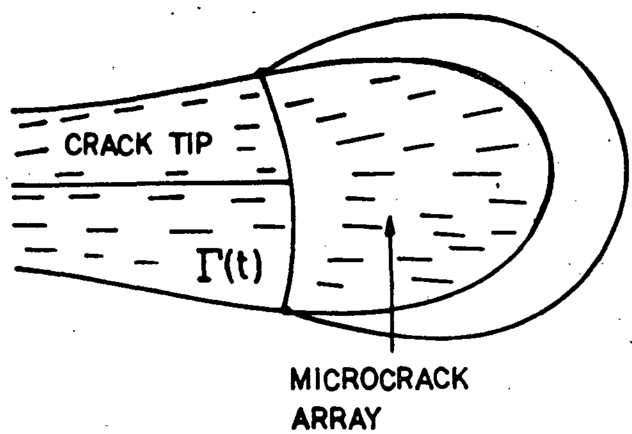

FIGURE 19

The energy release rate J preserves its path independent property for the model if evaluated on the contour surrounding the active zone.

Let us consider the problem of interaction of two cracks located on one line (i.e., Figure 5) in order to indicate in a simple

problem the particular path which leads to the solution of a general problem. The convenient path consists of the union of the three contours, Γ_0, Γ_1, and Γ_2 and the rectilinear segments connecting them, depicted in Figure 20. For the loops Γ_0, Γ_1, Γ_2 in Figure 20

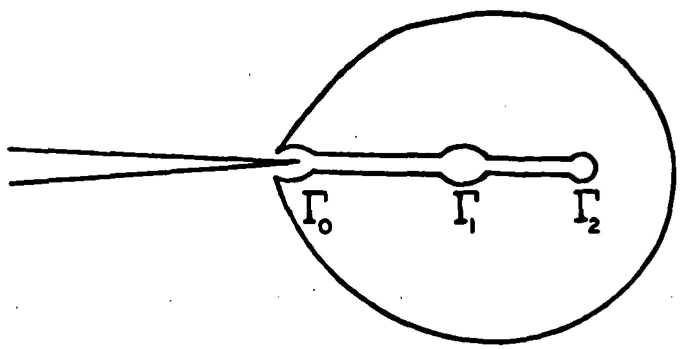

FIGURE 20

the integrals on rectilinear segments cancel and only the integrals on Γ_0, Γ_1, Γ_2, must be evaluated. J-integral evaluated on the loop around the crack tip is shown to be [30]:

$$J = \frac{K_1^2}{E} \tag{4.30}$$

where K_1 is stress intensity factor. thus, each of the integrals on Γ_0, Γ_1, Γ_2, can be calculated by means of (4.30).

This line of reasoning can be applied to the general problem of interaction of a macrocrack with a microcrack array (Figure 19). The energy release rate J for the general problem can be represented as follows:

$$J = J_o + \Delta J \tag{4.31}$$

where

$$J = \frac{(K_1^{eff})^2}{E} \tag{4.32}$$

is the energy release rate associated with the main crack and evaluated on the loop around the tip of the main crack Γ_0, (it should be noted that the microcrack array alsocontributes to J_0 through the use of K_1^{eff}) and

$$\Delta J = \sum_{i=1}^{N} \Delta J_i \tag{4.33}$$

is the energy release rate associated with the microcrack array. The parts of each ΔJ_i associated with two ends of the microcrack (i.e. on the countous similar to Γ_1 and Γ_2) enter with opposite signes (e.g., on the countour Γ_1 with (-) sign, and on the countour Γ_2 with (+) sign) because of the directions of the normals n_j to the contours. Fromt his remark follows that for piecewise constant appoximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$, each ΔJ_i vanishes because the stress intensity factors on both ends of each microcrack are equal.

In the general case of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ given by (4.7) the energy release rate associated with i-th microcrack of the array can be represented as follows:

$$\Delta J = \frac{1}{E} [(K_{10}^2 - K_{11}^2) + (K_{20}^2 - K_{21}^2)] \tag{4.34}$$

where K_{10} and K_{20} - are the mode I and mode II stress intensity factors at the microcrack tip which is closer to the main crack, respectively, K_{11} and K_{21} - are the mode I and mode II stress intensity factors at the microcrack tip which is farther from the main

crack tip. The second term in (4.34) appears because of the presence of the mode II loading on the microcracks [36].

In order to give the nontrivial example formula (4.34) has been used for evaluation of J-integral in the case of linear approximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ on each microcrack.

The stress intensity factor K_1 for a crack of length ℓ loaded by the normal stress $\sigma_{22}(x)$ is defined by the following expression:

$$K_1 = \frac{\ell}{\pi} \int_{-1}^{1} \sqrt{\frac{1 + X}{1 - X}}\, \sigma_{22}(X)dX \tag{4.35}$$

Analogous expression is valid for the loading by tangential stress $\sigma_{21}(x)$.

Under the assumption of linear approximation of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$ both of the above mentioned components are linear functions, and (4.35) can be rewritten as follows:

$$K_1 = \frac{\ell}{\pi} \int_{-1}^{1} \sqrt{\frac{1 + X}{1 - X}}\, (\alpha X + \beta)dX \tag{4.36}$$

where α and β are the constants of stress units, and x is the non-dimensional coordinate.

Integration yields

$$K_1 = \sqrt{\pi \ell} \quad (\tfrac{1}{2}\alpha + \beta) \tag{4.37}$$

The same formula is valid for K_2 but α and β would represent the

coefficients of linear function $\sigma_{21}(x)$.

Using the notation of (4.34), formula (4.37) gives K_{11}, while K_{10} is given by

$$K_{10} = \frac{\ell}{\pi}\int_{-1}^{1}\sqrt{\frac{1-X}{1+X}}\,(\alpha X+\beta)\,dX = \sqrt{\pi\ell}\,(-\tfrac{1}{2}\alpha+\beta) \tag{4.38}$$

Thus, ΔJ can be represented as follows:

$$\Delta J = \frac{\pi\ell}{E}[(\tfrac{1}{2}\alpha+\beta)^2 - (-\tfrac{1}{2}\alpha+\beta)^2] = \frac{2\pi\alpha\beta\ell}{E} \tag{4.39}$$

Using the results of the previous section, the coefficients α and β can be determined in terms of directional derivatives of the resulting stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$. formula (4.21), where $\{\underset{\sim}{b}\}$ given by (4.22) substituted into the righthand part, determines the traction and its derivatives at the centers of each microcrack of the array, i.e.,

$$\{\underset{\sim}{n}\cdot\underset{\sim}{\sigma}\} = [\{\underset{\sim}{H}\} + \{\underset{\sim}{S}\}\{\underset{\sim}{A}^{o}\}]\,\{\underset{\sim}{B}\}\{\underset{\sim}{t}\} - \{\underset{\sim}{t}\} \tag{4.40}$$

where $\underset{\sim}{n}$ is unit normal vector at the center of each microcrack. In (4.40) the matrix $\{\underset{\sim}{n}\cdot\underset{\sim}{\sigma}\}$ is determined by the values of dominating field traction and its directional derivatives at the center of microcracks $\{\underset{\sim}{t}\}$.

The coefficients $\alpha_k = \sigma'_{22}(x_o^k)$ and $\beta_k = \sigma_{22}(x\ )$ can be obtained from (4.40) (index k refers to the k-th microcrack). The same is

true for the shear mode. Substituting α_k and β_k into (4.39) we obtain ΔJ in a form:

$$\Delta J = \frac{2\Pi}{E} \sum_{i=1}^{N} [\sigma_{22}(X_o^i)\, \sigma'_{22}(X_o^i) + \sigma_{21}(X_o^i)\, \sigma'_{21}(X_o^i)]\,\ell^2 \qquad (4.41)$$

where the derivatives of the stress components are taken with respect to dimensional coordinate, and as a result in (4.41) appears the multiplier ℓ^2. Substitution of (4.41) into (4.31) gives the final formula for the energy release rate in the general problem.

In the particular problem of macro and microcrack located on one line (Figure 5) using (4.41) the expression for energy release rate can be written as follows:

$$J = \frac{(K_1^{eff})^2}{E} \left[1 - \frac{1}{2(1 + \delta/\ell)}\right] \qquad (4.42)$$

The expression above is approximated for $\delta/\ell > 0.25$. For smaller δ/ℓ higher order approximations have to be taken into account because of the strongly gradient field in the vicinity of the macrocrack tip.

CONCLUSION

1. The elastic stress field solution to the problem of interaction of a macrocrack with an array of arbitrarily located and oriented microcracks (within the limits of a smallscale model) has been obtained.

2. The elastic stress field solution for two and three crack interaction problems of interest to fracture mechanics has been obtained.

3. It has been shown that an array of microcracks in a main crack tip vicinity can either amplify the effective stress intensity factor or reduce it depending on the array's configuration.

4. The method leading to the solution of the general problem described above (i.e., the macrocrack-microcrack array interaction problem) refines the one suggested in the work [32].

5. Using the obtained elastic solution the energy release rate associated with the crack-layer translational motion has been evaluated.

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APPENDIX I

Derivation of the Second Green's Tensor for Plane Problem of Elastostatics

In this derivation of 2D Green's tensor we follow the routine of work [33] for obtaining of the second Green's tensor in 3D problem of elasticity.

The second Green's tensor can be defined by the following expression:

$$\underset{\sim}{n}_X \cdot \underset{\sim}{\sigma} = \underset{\sim}{\Phi}(\underset{\sim}{X}, \underset{\sim}{\xi}) \cdot \underset{\sim}{Q} \tag{1}$$

where $\underset{\sim}{Q}$ - is a unit force vector applied at point $\underset{\sim}{\xi}$ of an infinite elastic plane, $\underset{\sim}{\sigma}$ - stress tensor in the plane, and $\underset{\sim}{n}_X$ - is a unit normal vector at point x of the plane. Thus, influence tensor $\Phi(\underset{\sim}{x}, \underset{\sim}{\xi})$ has been defined as a linear operator which transforms the unit force $\underset{\sim}{Q}$ applied at some point into a traction vector at the other point. Surrounding the point with an imaginary closed contour we can write the equilibrium equation for the interior region:

$$\int_{\Gamma} \underset{\sim}{n} \cdot \underset{\sim}{\sigma} \, d\Gamma + \underset{\sim}{Q} = 0 \tag{2}$$

The value of integral in (2) does not depend on the contour of integration Γ, which may be chosen to be a circle. Then equation (2) taks the form,

$$\int_{\Gamma^*} \underset{\sim}{n} \cdot \underset{\sim}{\sigma}\, d\Gamma + \underset{\sim}{Q} = 0 \tag{3}$$

where Γ^* is a circle of unit radius where $\underset{\sim}{R} = \underset{\sim}{x} - \underset{\sim}{\xi}$, $|\underset{\sim}{R}| = R$. It follows from (3) that the magnitude of resultant force on any contour with a point inside of it does not depend on $\underset{\sim}{R}$ and always equals $-\underset{\sim}{Q}$. This is possible only if components of stress tensor $\underset{\sim}{\sigma}$ decrease as R^{-1}. Consequently, the components of the displacement vector u must have logarithmic singularity at point $\underset{\sim}{\xi}$. Therefore, in the Papkovich-Neuber representation for displacement vector $\underset{\sim}{u}$,

$$\underset{\sim}{u} = 4(1 - \nu)\ \underset{\sim}{B} - \mathrm{grad}(\underset{\sim}{R} \cdot \underset{\sim}{B}), \tag{4}$$

vector B has to be chosen as

$$\underset{\sim}{B} = \alpha\ \ell n\ R\ \underset{\sim}{Q} \tag{5}$$

where α is an unknown scalar constant, which can be determined from (2) after construction of a stress tensor $\underset{\sim}{\sigma}$ on the basis of (4). The introduction of the harmonic scalar into the Papkovich-Neuber representation is unnecessary in this problem. After substitution of (5) into (4) the displacement vector $\underset{\sim}{u}$ takes the form,

$$\underset{\sim}{u} = \alpha[(3 - 4\nu)\ \underset{\sim}{Q}\ell n\ R - \frac{\underset{\sim}{Q} \cdot \underset{\sim}{R}}{R^2}\ \underset{\sim}{R} \tag{6}$$

The strain tensor $\underset{\sim}{\varepsilon} = 1/2$ sym $\nabla\underset{\sim}{u}$, (where "sym" symbol refers to the operator of symmetrization, and ∇ operator is referred to 2D space) may be represented as follows:

$$\underset{\sim}{\varepsilon} = \alpha[(1 - 2\nu)\ \frac{\underset{\sim}{Q}\underset{\sim}{R} + \underset{\sim}{R}\underset{\sim}{Q}}{R^2} - \frac{\underset{\sim}{Q} \cdot \underset{\sim}{R}}{R^2}\ \underset{\sim}{E} + 2\frac{\underset{\sim}{Q} \cdot \underset{\sim}{R}}{R^4}\underset{\sim}{R}\underset{\sim}{R}] \tag{7}$$

where $\underset{\sim}{E}$ is the unit second rank tensor in 2D space. Using the strain tensor (7) and Hook's law for plane problem one can obtain stress tensor in a form:

$$\underset{\sim}{\sigma} = \frac{2\mu\alpha}{R}\left[(1 - 2\nu)\ (\underset{\sim}{Q}\underset{\sim}{R} + \underset{\sim}{R}\underset{\sim}{Q} - \underset{\sim}{Q}\cdot\underset{\sim}{R}\underset{\sim}{E}) + 2\frac{\underset{\sim}{Q}\cdot\underset{\sim}{R}}{R^2}\underset{\sim}{R}\underset{\sim}{R}\right] \tag{8}$$

Then, equation (2) can be rewritten as follows:

$$\frac{2\mu\alpha}{R^2}\int_{\Gamma^*}\left[(1 - 2\nu)(\underset{\sim}{n}\cdot\underset{\sim}{Q}\underset{\sim}{R} + \underset{\sim}{n}\cdot\underset{\sim}{R}\underset{\sim}{Q} - \underset{\sim}{n}\underset{\sim}{Q}\cdot\underset{\sim}{R}) + 2\frac{\underset{\sim}{Q}\cdot\underset{\sim}{R}}{}\underset{\sim}{n}\cdot\underset{\sim}{R}\underset{\sim}{R}\right]d\Gamma^* + \underset{\sim}{Q} = 0 \tag{9}$$

Taking into account that $R\,\underset{\sim}{n} = \underset{\sim}{R}$ on the circle Γ^* the following relations hold

$$\underset{\sim}{n}\cdot\underset{\sim}{Q}\underset{\sim}{R} - \underset{\sim}{n}\underset{\sim}{Q}\cdot\underset{\sim}{R} = 0, \quad \text{and} \quad \frac{\underset{\sim}{Q}\cdot\underset{\sim}{R}}{R^2}\,\underset{\sim}{n}\cdot\underset{\sim}{R}\underset{\sim}{R} = \underset{\sim}{Q}\cdot\underset{\sim}{R}\,\underset{\sim}{n} \tag{10}$$

In view of (10), (9) takes form,

$$\frac{2\mu\alpha}{R^2}\int_{\Gamma^*}\left[(1 - 2\nu)R\underset{\sim}{Q} + 2\underset{\sim}{Q}\cdot\underset{\sim}{R}\underset{\sim}{n}\right]d\Gamma^* + \underset{\sim}{Q} = 0 \tag{9a}$$

Both integrals in (9a) yield

$$\int_{\Gamma^*} R\underset{\sim}{Q}\ d\Gamma^* = \int_{\Gamma^*} \underset{\sim}{Q}\cdot\underset{\sim}{R}\underset{\sim}{n}\ d\Gamma^* = 2\pi R^2\underset{\sim}{Q} \tag{10}$$

Formula (9a) with integrals given by (10) results in

$$\alpha = -\frac{1}{8\mu\pi(1 - \nu)} \tag{11}$$

Substituting (11) into (6) we obtain

$$u = \frac{1}{8\mu\pi(1 - \nu)} [(3 - 4\nu)\ln R \cdot \underset{\sim}{Q} - \frac{R}{R^2} \underset{\sim}{R} \cdot \underset{\sim}{Q}]$$

From the last equation, it follows that the first Green's tensor (i.e., Kelvin-Somigliana tensor) defined by the relation $\underset{\sim}{u} = \underset{\sim}{U}.\underset{\sim}{Q}$ can be represented as follows:

$$\underset{\sim}{U} = \frac{1}{8\pi\mu(1 - \nu)} [(3 - 4\nu)\ln R \cdot \underset{\sim}{E} - \frac{\underset{\sim}{R}\underset{\sim}{R}}{R^2}] \tag{12}$$

Substitution of (11) into (8) gives traction vector at point $\underset{\sim}{x}$ in a form,

$$\underset{\sim}{n}_X \cdot \underset{\sim}{\sigma} = \frac{1}{4\pi(1 - \nu)R^2} [(1 - 2\nu)(\underset{\sim}{n}_X \underset{\sim}{R} \cdot \underset{\sim}{Q} - \underset{\sim}{n}_X \cdot \underset{\sim}{R}\underset{\sim}{Q} - \underset{\sim}{R}\underset{\sim}{n}_X \cdot \underset{\sim}{Q}) - 2\frac{\underset{\sim}{n}_X \cdot \underset{\sim}{R}}{R^2}\underset{\sim}{R}\underset{\sim}{R} \cdot \underset{\sim}{Q}] \tag{13}$$

From the last formula the second Green's tensor can be obtained as follows:

$$\underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{X}) = \frac{1}{4\pi R^2(1 - \nu)}[(1 - 2\nu)(\underset{\sim}{n}_X \underset{\sim}{R} - \underset{\sim}{R}\underset{\sim}{n}_X - \underset{\sim}{n}_X \underset{\sim}{R}\underset{\sim}{E}) - 2\frac{\underset{\sim}{n}_X \cdot \underset{\sim}{R}}{R^2}\underset{\sim}{R}\underset{\sim}{R}] \tag{14}$$

It should be noted that $\Phi(\underset{\sim}{x}, \underset{\sim}{\xi})$ defined by (14) can be used in the representation of displacement vector $\underset{\sim}{u}$ in terms of double layer potential as follows:

$$\underset{\sim}{u}(\underset{\sim}{X}) = \int \underset{\sim}{b}(\underset{\sim}{\xi}) \cdot \underset{\sim}{\Phi}(\underset{\sim}{\xi}, \underset{\sim}{X})d\underset{\sim}{\xi}$$

where $\underset{\sim}{\xi}$, and $\underset{\sim}{x}$ change places and $\underset{\sim}{b}(\underset{\sim}{x})$ is double layer potential density [33].

APPENDIX II

Evaluation of The Integrals

1.
$$I_o = \int_{c-\ell}^{c+\ell} \frac{\sqrt{-\xi^2 + 2c\xi - (c^2 - \ell^2)}}{(\xi - X)^2} d\xi, \quad X \overline{\in} (c - \ell, c + \ell)$$

Substituting $t = \xi - x$, the integral takes the form

$$I_o = -\int_{X-(c-\ell)}^{X-(c+\ell)} \frac{\sqrt{\ell^2 - (X-t-c)^2}}{t^2} dt = -\{\frac{\sqrt{T}}{t} + \int_{X-(c-\ell)}^{X-(c+\ell)} \frac{dt}{\sqrt{T}} + (X-c) \int_{X-(C-\ell)}^{X-(C+\ell)} \frac{dt}{t\sqrt{T}}\}$$

where $T = \ell^2 - (x - c - t)^2$

The integral I_0 in the last expression may be found in [37]; the result of integration is

$$I_o = \frac{\pi}{\ell}[\frac{X - c}{\sqrt{(X - C)^2 - \ell^2}} - 1]$$

2. Analogously

$$I_1 = \int_{c-\ell}^{c+\ell} \frac{\sqrt{-\xi^2 + 2c\xi - (c^2 - \ell^2)}}{(\xi - X)} d\xi = -\int_{x-(c-\ell)}^{x-(c+\ell)} \frac{\sqrt{\ell^2 - (X-t-c)^2}}{t} dt = \pi(X-c)[1 - (\frac{\ell}{X-c})^2 - 1]$$

$X^{-}(c-\ell, c+\ell)$

3. The integral represented by formula (2.30) Chapter II, can be estimated using the mean value theorem as follows

$$\int_{-\ell}^{\ell} e(\xi') \frac{3X_2'^4 - (\xi - X_1')^4 - 6(\xi - X_1')^2 X_2'^2}{[(\xi - X_1')^2 + X_2'^2]^3} d\xi =$$

$$e(\xi_o') \int_{-\ell}^{\ell} \frac{3X_2'^4 - (\xi - X_1')^4 - 6(\xi - X_1')^2 X_2'^2}{[(\xi - X_1')^2 + X_2'^2]^3} d\xi ,$$

where $\xi_0' \in (-\ell, \ell)$

Each of three integrals

$$\int_{-\ell}^{\ell} \frac{d\xi}{[(\xi - X_1')^2 + X_2'^2]^3} , \qquad \int_{-\ell}^{\ell} \frac{(\xi - X_1')^4 \, d\xi}{[(\xi - X_1')^2 + X_2'^2]^3} ,$$

$$\int_{-\ell}^{\ell} \frac{(\xi - X_1')^2 \, d\xi}{[(\xi - X_1')^2 + X_2'^2]^3}$$

are tabulated in [37] and the integration results in formula (2.34) Chapter II.

4. Substitution of formula (2.36) into (2.33), Chapter II, leads to the integral

$$\int_{-\ell}^{\ell} \sqrt{\frac{\ell_o + X}{\ell_o - X}} \left[\frac{1}{\ell_o - X - \ell} - \frac{1}{\ell_o - X + \ell}\right] dX$$

which can be transformed into the integal

$$2\int_{0}^{\frac{\pi}{2}}\left[\frac{1+\cos 2t}{(1-\ell')-\cos 2t}-\frac{1+\cos 2t}{(1+\ell')-\cos 2t}\right]dt,$$

where $\ell' = \ell/\ell_0$, by means of nondimensionalization and subsequent substitute $1 - X = 2\sin^2 t$. The last two integrals are tabulated in [37].

APPENDIX III

Evaluation of The Influence Function $F(\ell, x)$

The influence function $F(\ell, x)$ is given by formula (2.29):

$$F(\ell, \underset{\sim}{X}) = \frac{4\ell}{E}\underset{\sim}{n}(\underset{\sim}{X})\underset{\sim}{n}(\underset{\sim}{X})\underset{\sim}{T}_x \cdot \int_\ell e(\xi)\underset{\sim}{n}(\xi)\cdot\underset{\sim}{\sigma}(\xi)\cdot\underset{\sim}{\Phi}(\xi, \underset{\sim}{X})d\xi$$

The operator $\underset{\sim}{n}(\underset{\sim}{x})\ \underset{\sim}{n}(\underset{\sim}{x}).\underset{\sim}{T}_x$ applied to the displacement field produces σ_{22} component of the stress field. In order to evaluate the σ_{22} component of the stress field by means of (2.6), formulas (2.7) and (2.8) must be used. Taking into account that

$$\underset{\sim}{n}(\xi)\cdot\underset{\sim}{\sigma}(\xi)\cdot\underset{\sim}{\Phi}(\xi, \underset{\sim}{X}) = n_j(\xi)\sigma_{ij}(\xi)\ \underset{\sim}{\Phi}_{iK}(\xi, \underset{\sim}{X}) = \sigma_{22}\ \Phi_{2K}$$

(because $n_1(\xi) = 0$, $n_2(\xi) = 1$, and $\sigma_{12} = 0$ for this problem) and substituting the last expression into (2.7), with Φ_{ik} given by (2.8), the $F(\ell, x)$ with the help of (2.6) takes the form

$$F(\ell, X) = \ell \int_\ell e(\xi)\ \{ -\frac{2}{R^4}\ [\nu(X_1 - \xi)^2 + \frac{\nu(3 + 2\nu)}{1 - 2\nu}\ X_2^2] + $$
$$+ \frac{\nu}{1-2\nu} \cdot \frac{8}{R^6}\ X_2^2[X_2^2 + (X_1 - \xi)^2] - \frac{1-2\nu}{R^2} - \frac{4(1 + \nu)}{R^4}\ X_2^2 + \frac{8X_2^4}{R^6}\ \}\ d\xi$$

where

$$R^2 = (X_1 - \xi)^2 + X_2^2$$

The last equation reduces to (2.3) by factoring out the $1/R^6$.

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15. Supplementary Notes

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16. Abstract

This work considers the problem of elastic interaction of a macrocrack with an array of microcracks in the vicinity of the macrocrack tip. Using the double layer potential techniques, the solution to the problem within the framework of the plane problem of elastostatics has been obtained. Three particular problems of interest to fracture mechanics have been analyzed. It follows from analysis that microcrack array can either amplify or reduce the resulting stress field of the macrocrack-microcrack array system depending on the array's configuration. Using the obtained elastic solution the energy release rate associated with the translational motion of the macrocrack-microcrack array system has been evaluated.

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